§1.1.1 Exercise solutions (manuscript by R. Thomas, printed by R. Craigen)

#2 To show $P(n)$: $3+3^2+\cdots+3^n = \frac{3^{n+1}-3}{2}$ for all $n \geq 1$, we begin by verifying $P(1)$: $3^1 = \frac{3^2-3}{2}$, which we see is true when we write it. To show $P(k)$: $3+3^2+\cdots+3^k = \frac{3^{k+1}-3}{2}$ implies $P(k+1)$: $3+3^2+\cdots+3^{k+1} = \frac{3^{k+2}-3}{2}$, we need to find the left side of $P(k)$ in the left side of $P(k+1)$.

$$3+3^2+\cdots+3^k+3^{k+1} \overset{\text{by } P(k)}{=} \frac{3^{k+1}-3}{2}+3^{k+1} = \frac{3\cdot 3^{k+1}-3}{2} = \frac{3^{k+2}-3}{2}.$$

By the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

$\boxed{1.1.1:\ 13}$ (Prove that) $1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1) = \frac{n(n+1)(n+2)}{3}$.

CLAIM: $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$ for all $n \geq 1$.

PROOF: $\Sigma_{k=1}^{1} k(k+1) = 1\cdot 2 = 2 = \frac{1(1+1)(1+2)}{3}$, so the claim is true for $n=1$.

Now suppose the claim is true for $n=m$. That is, $\Sigma_{k=1}^{m} k(k+1) = \frac{m(m+1)(m+2)}{3}$. Then

$$\begin{aligned}\Sigma_{k=1}^{m+1} k(k+1) &= \Sigma_{k=1}^{m} k(k+1)+(m+1)(m+2)\\ &= \frac{m(m+1)(m+2)}{3}+(m+1)(m+2)\text{(by our inductive hypothesis)}\\ &= (m+1)(m+2)\left[\frac{m}{3}+1\right] = \frac{(m+1)(m+2)(m+3)}{3}.\end{aligned}$$

Thus the claim is true for $n=m+1$.

Therefore, by (mathematical) induction, the claim is true for all $n \geq 1$. □

Trim #23

To show $576 = 24^2$ divides evenly $5^{2n+2} - 24n - 25$ for all $n \geq 1$ by induction we need to verify $P(1): 576 \mid 5^4 - 24 - 25$, which is itself 576. Then use $P(k): 576 \mid 5^{2k+2} - 24k - 25$ to show $P(k+1): 576 \mid 5^{2k+4} - 24(k+1) - 25$.

$$
\begin{aligned}
5^{2k+4} - 24(k+1) - 25 &= 5^2 \cdot 5^{2k+2} - 24k - 49 \\
&= 5^2(5^{2k+2} - 24k - 25) + 5^2 \cdot 24k + 625 - 24k - 49 \\
&= 5^2(5^{2k+2} - 24k - 25) + 576k + 576.
\end{aligned}
$$

As 576 divides the three terms above, it divides their sum.

Now the principle of mathematical induction shows that $P(n)$ is true for all $n \geq 1$.

Trim #25

To show $(x+y) \mid (x^{2n+1} + y^{2n+1})$ for all $n \geq 0$, we need to examine the case $n = 0$. Then $x^{2n+1} + y^{2n+1} = x + y$, which is certainly a multiple of $x+y$. $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$. $P(k)$ says $(x+y) \mid (x^{2k+1} + y^{2k+1})$. $P(k+1)$ says $(x+y) \mid (x^{2k+3} + y^{2k+3})$. There are several ways to break up $x^{2k+3} + y^{2k+3}$ to see ~~that~~ it is divisible by $x+y$.

(a) $y^2(x^{2k+1} + y^{2k+1}) - y^2 x^{2k+1} + x^2 x^{2k+1} = y^2(x^{2k+1} + y^{2k+1}) + (x^2 - y^2)x^{2k+1}$.

(b) $x^2(x^{2k+1} + y^{2k+1}) - x^2 y^{2k+1} + y^2 y^{2k+1}$

$$
\begin{aligned}
&= x^2(x^{2k+1} + y^{2k+1}) - x^2 y^{2k+1} + y^2(x^{2k+1} + y^{2k+1}) - y^2 x^{2k+1} \\
&= (x^2 + y^2)(x^{2k+1} + y^{2k+1}) - x^2(x^{2k+1} + y^{2k+1}) + x^2 x^{2k+1} - y^2 x^{2k+1} \\
&= y^2(x^{2k+1} + y^{2k+1}) + (x^2 - y^2)x^{2k+1}.
\end{aligned}
$$

(c) $(x+y)(x^{2k+2} + y^{2k+2}) - yx^{2k+2} - xy^{2k+2}$

$$= (x+y)(x^{2k+2} + y^{2k+2}) - xy(x^{2k+1} + y^{2k+1}).$$

(d) $x^2(x^{2k+1} + y^{2k+1}) + y^2(x^{2k+1} + y^{2k+1}) - (x^2 y^{2k+1} + y^2 x^{2k+1})$

(as in b. line 2)

$$= (x^2 + y^2)(x^{2k+1} + y^{2k+1}) - x^2 y^2(x^{2k-1} + y^{2k-1}).$$

In each case each term in the final line is divisible by $x+y$ either always or by the induction assumption, but in (d) one is using strong induction because of needing $P(k-1)$: $(x+y) \mid (x^{2k-1}+y^{2k-1})$. The need for strong induction usually arises from the argument.

The proof by mathematical induction is complete

$\boxed{1.1.1:\ 27}$ [NOTE: I don't like the use of "verify" here, which usually means "check that this works out", usually asking for you to do a calculation. What they mean is "prove that it is true (using mathematical induction)". "Verify" is used because, presumably, you *know* this formula but don't necessarily remember why it is true. In the proof we assume the formula for a triangle is known, i.e., the sum of the interior angles of a triangle is 180°, or π radians. You might want to prove this (it's not hard) but that is a matter of elementary geometry, not induction, so it is of little value here. Also note two twists on the usual form of mathematical induction: (i) a base case that is not $n = 1$, and (ii) an inductive hypothesis that is more powerful than the usual kind.]

PROOF: Observe that the statement makes sense only for $n \geq 3$; we shall prove it for all such n.

Base Case: When $n = 3$ such a polygon is a triangle; it is known that the sum of the interior angles of a triangle is $180° = (3-2)\pi^R$, so the assertion is true in this case.

Inductive Step: [NOTE: I'm too lazy to insert a picture here so I'll describe what to do in words so you'll have to draw your own. In an assignment or exam include the picture if it is critical to your proof!]

Inductive Hypothesis: Suppose the sum of the interior angles of any polygon with h sides, where $h \leq k$, is $(h-2)\pi$.

[NOTE: I am using a type of I.H. here that is sometimes called "complete induction" – you assume the claim is true for ALL values up to k. It's not hard to see that this is equivalent to ordinary induction, but it is more convenient here as you'll see. A minor modifiction of this proof can be made so that you only need to assume the truth of the claim for $n = k$.]

Let P be a polygon with $k+1$ sides. A chord joining two vertrices can always be added to a polygon with at least 4 sides. Since $k+1 > 3$, P has such a chord, $\overline{xy}$.

This chord cuts P into two smaller polygons, A and B, having a and b sides, respectively, where $a, b \geq 3$, $a+b = k+3$ (because vertices x and y are used in both A and B). Therefore $a, b \leq k$.

By our inductive hypothesis, the sum of the interior angles of A is $(a-2)\pi$ and for B it is $(b-2)\pi$. The sum of the interior angles of P is the sum of these two numbers [NOTE: your diagram will make this obvious!], $(a+b-4)\pi = ((k+3)-4)\pi$. So the sum of the interior angles of any polygon of $k+1$ sides is $((k+1)-2)\pi$.

This completes the inductive step.

Therefore, by mathematical induction, the sum of the interior angles of any polygon having n sides is $(n-2)\pi$. □

$\boxed{1.1.1{:}\ 45}$ (Prove that) $2n + (2n+1) + (2n+2) + \cdots + 5n = \frac{7n(3n+1)}{2}$.

PROOF: For all $n \geq 1$, let P_n be the statement $\displaystyle\sum_{k=0}^{3n} 2n + k = \frac{7n(3n+1)}{2}$.

CLAIM: P_n is true, for all $n \geq 1$.

Since $2 + 3 + 4 + 5 = 14 = \frac{7\cdot 1(3\cdot 1+1)}{2}$, P_1 is true.

Suppose P_m is true for some $m \geq 1$. That is, $\displaystyle\sum_{k=0}^{3m} 2n + k = \frac{7m(3m+1)}{2}$.

Then

$$\begin{aligned}\sum_{k=0}^{3(m+1)} 2(m+1) + k &= \sum_{k=0}^{3m}(2m+k) + (5m+1) + (5m+2) + (5m+3)\\ &\quad + (5m+4) + (5m+5) - (2m) - (2m+1)\\ &= \frac{7m(3m+1)}{2} + 21m + 14 \text{ (By inductive hypothesis)}\\ &= \frac{21m^2 + 28m + 28}{2} = \frac{7(m+1)(3(m+1)+1)}{2},\end{aligned}$$

so P_{m+1} follows. That is, $P_m \implies P_{m+1}$.

Therefore, by induction, P_n holds for all $n \geq 1$, as claimed. $\square$

$\boxed{1.1.1{:}\ 47}$ Verify that, for $n \geq 1$, $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \leq 3 - \frac{1}{n}$

PROOF: Since $1 + \frac{1}{1!} = 2 \leq 3 - \frac{1}{1}$, the claim is true for $n = 1$.

Suppose that the claim is true for $n = k$.

That is, $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} \leq 3 - \frac{1}{k}$. Then

$$\begin{aligned}&1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(k+1)!} = \left[1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}\right] + \frac{1}{(k+1)!}\\ &\leq 3 - \frac{1}{k} + \frac{1}{(k+1)!} \leq 3 - \frac{1}{k+1},\end{aligned}$$

the last inequality being justified as follows: $k! \geq k$, so $\frac{1}{k!} \leq \frac{1}{k}$, so $1 + \frac{1}{k} - \frac{1}{k!} \geq 1$, so $\frac{k+1}{k} - \frac{1}{k!} \geq 1$, so $\frac{1}{k} - \frac{1}{(k+1)!} \geq \frac{1}{k+1}$, so $3 - \frac{1}{k} + \frac{1}{(k+1)!} \leq 3 - \frac{1}{k+1}$.

It follows that the claim is true for $n = k + 1$.

By induction, the claim is true for all $n \geq 1$. $\square$

$\boxed{1.1.1{:}\ 35}$ Suppose n people stand in line at a ticket counter. Show that if the first person in line is a woman and the last person is a man, then somewhere in the line, there must be a man immediately behind a woman.

[NOTE: the hardest part of this problem is resisting the effort to say, "it's obviously true", or producing a different kind of argument. Also note that the base case cannot be $n = 1$.]

PROOF: Suppose $n = 2$. Then, as described, the line necessarily consists of a woman, and a man immediately behind her, so the claim is true in this case.

Suppose the statement is true for lines of up to $k(\geq 2)$ people.

[NOTE: once more we use a "complete induction" hypothesis!]

Let L be a line of $k + 1$ people. Since L has at least 3 people, it includes a person, A, that is neither first nor last in line. If A is a woman, let L' be the portion of line L that begins with A. If A is a man, let L' be the portion of the line that ends with A. In either case, L' has $\leq k$ people, begins with a woman and ends with a man. By our inductive hypothesis, L' has a man immediately behind a woman. Since L' is a portion of line L, this is also true of line L.

By induction, the statement is true for all $n \geq 2$. □

Trim #41

The solution written out here doesn't replace n with k and k+1. It illustrates the place where substitution of n+1 for n occurs, an operation better and easily avoided. This $P(n+1)$ is trickier than some.

The statements of $P(n)$ and $P(n+1)$ are always important, not always as easy as above. Here $P(n)$ is $n+(n+1)+\cdots+2n = \frac{3n(n+1)}{2}$ and $P(n+1)$ is $(n+1)+(n+2)+\cdots+2(n+1) = \frac{3(n+1)(n+2)}{2}$. $P(1)$ is $1+2 = \frac{3\cdot 1\cdot 2}{2}$, obviously true when stated. To show $P(n) \Rightarrow P(n+1)$ we need to state them and then see how their left sides are related

$$(n+1)+(n+2)+\cdots+2n+(2n+1)+(2n+2) = n+(n+1)+\cdots+2n - n + (2n+1)+(2n+2)$$

by $P(n)$
$$= \frac{3(n)(n+1)}{2} + 3n + 3$$
$$= \frac{3(n+1)n + 2\cdot 3(n+1)}{2}$$
$$= \frac{3(n+1)(n+2)}{2}.$$

The proof by mathematical induction is complete.