In Exercises 1-15 use mathematical induction to establish the formula for $n \geq 1$.

1. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Proof:

For $n = 1$, the statement reduces to $1^2 = \dfrac{1 \cdot 2 \cdot 3}{6}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}\,, \tag{1}$$

we will prove that the statement must be true for $n = k + 1$:

$$1^2 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}\,. \tag{2}$$

The left-hand side of (2) can be written as

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2.$$

In view of (1), this simplifies to:

$$\begin{aligned}
\left(1^2 + 2^2 + 3^2 + \cdots + k^2\right) + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

Thus the left-hand side of (2) is equal to the right-hand side of (2). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

2. $3 + 3^2 + 3^3 + \cdots + 3^n = \dfrac{3^{n+1} - 3}{2}$

Proof:

For $n = 1$, the statement reduces to $3 = \dfrac{3^2 - 3}{2}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$3 + 3^2 + 3^3 + \cdots + 3^k = \frac{3^{k+1} - 3}{2}\,, \tag{3}$$

we will prove that the statement must be true for $n = k + 1$:

$$3 + 3^2 + 3^3 + \cdots + 3^{k+1} = \frac{3^{k+2} - 3}{2}\,. \tag{4}$$

The left-hand side of (4) can be written as

$$3 + 3^2 + 3^3 + \cdots + 3^k + 3^{k+1}.$$

In view of (3), this simplifies to:

$$\begin{aligned}\left(3 + 3^2 + 3^3 + \cdots + 3^k\right) + 3^{k+1} &= \frac{3^{k+1} - 3}{2} + 3^{k+1} \\ &= \frac{3^{k+1} - 3 + 2 \cdot 3^{k+1}}{2} \\ &= \frac{3 \cdot 3^{k+1} - 3}{2} \\ &= \frac{3^{k+2} - 3}{2}.\end{aligned}$$

Thus the left-hand side of (4) is equal to the right-hand side of (4). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

Proof:

For $n = 1$, the statement reduces to $1^3 = \dfrac{1^2 \cdot 2^2}{4}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}, \tag{5}$$

we will prove that the statement must be true for $n = k + 1$:

$$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}. \tag{6}$$

The left-hand side of (6) can be written as

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3.$$

In view of (5), this simplifies to:

$$\begin{aligned}\left(1^3 + 2^3 + 3^3 + \cdots + k^3\right) + (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4}.\end{aligned}$$

Thus the left-hand side of (6) is equal to the right-hand side of (6). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

4. $1+3+6+10+\cdots+\dfrac{n(n+1)}{2}=\dfrac{n(n+1)(n+2)}{6}$

Proof:

For $n=1$, the statement reduces to $1=\dfrac{1\cdot 2\cdot 3}{6}$ and is obviously true.

Assuming the statement is true for $n=k$:

$$1+3+6+10+\cdots+\frac{k(k+1)}{2}=\frac{k(k+1)(k+2)}{6}, \tag{7}$$

we will prove that the statement must be true for $n=k+1$:

$$1+3+6+10+\cdots+\frac{(k+1)(k+2)}{2}=\frac{(k+1)(k+2)(k+3)}{6}. \tag{8}$$

The left-hand side of (8) can be written as

$$1+3+6+10+\cdots+\frac{k(k+1)}{2}+\frac{(k+1)(k+2)}{2}.$$

In view of (7), this simplifies to:

$$\begin{aligned}\left[1+3+6+10+\cdots+\frac{k(k+1)}{2}\right]&+\frac{(k+1)(k+2)}{2}\\ &=\frac{k(k+1)(k+2)}{6}+\frac{(k+1)(k+2)}{2}\\ &=\frac{k(k+1)(k+2)+3(k+1)(k+2)}{6}\\ &=\frac{(k+1)(k+2)(k+3)}{6}.\end{aligned}$$

Thus the left-hand side of (8) is equal to the right-hand side of (8). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

5. $1+4+7+\cdots+(3n-2)=\dfrac{n(3n-1)}{2}$

Proof:

For $n=1$, the statement reduces to $1=\dfrac{1\cdot 2}{2}$ and is obviously true.

Assuming the statement is true for $n=k$:

$$1+4+7+\cdots+(3k-2)=\frac{k(3k-1)}{2}, \tag{9}$$

we will prove that the statement must be true for $n=k+1$:

$$1+4+7+\cdots+[3(k+1)-2]=\frac{(k+1)[3(k+1)-1]}{2}. \tag{10}$$

The left-hand side of (10) can be written as

$$1 + 4 + 7 + \cdots + (3k - 2) + [3(k + 1) - 2].$$

In view of (9), this simplifies to:

$$\begin{aligned}[1 + 4 + 7 + \cdots + (3k - 2)] + (3k + 1) &= \frac{k(3k - 1)}{2} + (3k + 1) \\ &= \frac{k(3k - 1) + 2(3k + 1)}{2} \\ &= \frac{3k^2 + 5k + 2}{2} \\ &= \frac{(k + 1)(3k + 2)}{2}.\end{aligned}$$

The last expression is obviously equal to the right-hand side of (10). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

6. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \dfrac{n(2n - 1)(2n + 1)}{3}$

Proof:

For $n = 1$, the statement reduces to $1^2 = \dfrac{1 \cdot 3 \cdot 3}{3}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$1^2 + 3^2 + 5^2 + \cdots + (2k - 1)^2 = \frac{k(2k - 1)(2k + 1)}{3}, \tag{11}$$

we will prove that the statement must be true for $n = k + 1$:

$$1^2 + 3^2 + 5^2 + \cdots + [2(k + 1) - 1]^2 = \frac{(k + 1)[2(k + 1) - 1][2(k + 1) + 1]}{3}. \tag{12}$$

The left-hand side of (12) can be written as

$$1^2 + 3^2 + 5^2 + \cdots + (2k - 1)^2 + [2(k + 1) - 1]^2.$$

In view of (11), this simplifies to:

$$
\begin{aligned}
\left[1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2\right] + (2k+1)^2 \\
&= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \\
&= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3} \\
&= \frac{(2k+1)[k(2k-1) + 3(2k+1)]}{3} \\
&= \frac{(2k+1)(2k^2+5k+3)}{3} \\
&= \frac{(2k+1)(k+1)(2k+3)}{3} \\
&= \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}.
\end{aligned}
$$

Thus the left-hand side of (12) is equal to the right-hand side of (12). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

7. $1 + 5 + 9 + 13 + \cdots + (4n-3) = 2n^2 - n$

Proof:

For $n = 1$, the statement reduces to $1 = 2 \cdot 1^2 - 1$ and is obviously true.

Assuming the statement is true for $n = k$:

$$1 + 5 + 9 + 13 + \cdots + (4k-3) = 2k^2 - k\,, \tag{13}$$

we will prove that the statement must be true for $n = k+1$:

$$1 + 5 + 9 + 13 + \cdots + [4(k+1) - 3] = 2(k+1)^2 - (k+1)\,. \tag{14}$$

The left-hand side of (14) can be written as

$$1 + 5 + 9 + 13 + \cdots + (4k-3) + [4(k+1) - 3].$$

In view of (13), this simplifies to:

$$
\begin{aligned}
[1 + 5 + 9 + 13 \cdots + (4k-3)] + (4k+1) &= (2k^2 - k) + (4k+1) \\
&= 2k^2 + 3k + 1 = (k+1)(2k+1) \\
&= (k+1)[2(k+1) - 1] \\
&= 2(k+1)^2 - (k+1).
\end{aligned}
$$

Thus the left-hand side of (14) is equal to the right-hand side of (14). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

8. $2 + 2^3 + 2^5 + \cdots + 2^{2n-1} = \dfrac{2(2^{2n} - 1)}{3}$

Proof:

For $n = 1$, the statement reduces to $2 = \dfrac{2(2^2 - 1)}{3}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$2 + 2^3 + 2^5 + \cdots + 2^{2k-1} = \frac{2(2^{2k} - 1)}{3}, \tag{15}$$

we will prove that the statement must be true for $n = k + 1$:

$$2 + 2^3 + 2^5 + \cdots + 2^{2(k+1)-1} = \frac{2(2^{2(k+1)} - 1)}{3}. \tag{16}$$

The left-hand side of (16) can be written as

$$2 + 2^3 + 2^5 + \cdots + 2^{2k-1} + 2^{2(k+1)-1}.$$

In view of (15), this simplifies to:

$$\begin{aligned}
\left(2 + 2^3 + 2^5 + \cdots + 2^{2k-1}\right) + 2^{2k+1} &= \frac{2(2^{2k} - 1)}{3} + 2^{2k+1} \\
&= \frac{2(2^{2k} - 1) + 3 \cdot 2^{2k+1}}{3} \\
&= \frac{2^{2k+1} - 2 + 3 \cdot 2^{2k+1}}{3} \\
&= \frac{4 \cdot 2^{2k+1} - 2}{3} \\
&= \frac{2(2 \cdot 2^{2k+1} - 1)}{3} \\
&= \frac{2(2^{2k+2} - 1)}{3}.
\end{aligned}$$

The last expression is obviously equal to the right-hand side of (16). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

9. $\dfrac{1}{1 \cdot 2 \cdot 3} + \dfrac{1}{2 \cdot 3 \cdot 4} + \dfrac{1}{3 \cdot 4 \cdot 5} + \cdots + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$

Proof:

For $n = 1$, the statement reduces to $\dfrac{1}{1 \cdot 2 \cdot 3} = \dfrac{1 \cdot 4}{4 \cdot 2 \cdot 3}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}, \tag{17}$$

we will prove that the statement must be true for $n = k + 1$:

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots+\frac{1}{(k+1)(k+2)(k+3)}=\frac{(k+1)(k+4)}{4(k+2)(k+3)}\,. \tag{18}$$

The left-hand side of (18) can be written as

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots+\frac{1}{k(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}.$$

In view of (17), this simplifies to:

$$\begin{aligned}
\left[\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots+\frac{1}{k(k+1)(k+2)}\right]&+\frac{1}{(k+1)(k+2)(k+3)}\\
&=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)}\\
&=\frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}\\
&=\frac{k(k^2+6k+9)+4}{4(k+1)(k+2)(k+3)}\\
&=\frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}\\
&=\frac{(k+1)(k^2+5k+4)}{4(k+1)(k+2)(k+3)}\\
&=\frac{k^2+5k+4}{4(k+2)(k+3)}\\
&=\frac{(k+1)(k+4)}{4(k+2)(k+3)}.
\end{aligned}$$

Thus the left-hand side of (18) is equal to the right-hand side of (18). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

10. $4^3 + 8^3 + 12^3 + \cdots + (4n)^3 = 16n^2(n + 1)^2$

Proof:

For $n = 1$, the statement reduces to $4^3 = 16 \cdot 1^2 \cdot 2^2$ and is obviously true.

Assuming the statement is true for $n = k$:

$$4^3 + 8^3 + 12^3 + \cdots + (4k)^3 = 16k^2(k + 1)^2\,, \tag{19}$$

we will prove that the statement must be true for $n = k + 1$:

$$4^3 + 8^3 + 12^3 + \cdots + [4(k + 1)]^3 = 16(k + 1)^2(k + 2)^2\,. \tag{20}$$

The left-hand side of (20) can be written as

$$4^3 + 8^3 + 12^3 + \cdots + (4k)^3 + [4(k + 1)]^3.$$

In view of (19), this simplifies to:

$$\begin{aligned}\left[4^3+8^3+12^3+\cdots+(4k)^3\right]+4^3(k+1)^3 &= 16k^2(k+1)^2+64(k+1)^3\\ &= 16(k+1)^2[k^2+4(k+1)]\\ &= 16(k+1)^2(k+2)^2.\end{aligned}$$

Thus the left-hand side of (20) is equal to the right-hand side of (20). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

11. $\dfrac{1}{5^2}+\dfrac{1}{5^4}+\cdots+\dfrac{1}{5^{2n}}=\dfrac{1}{24}\left(1-\dfrac{1}{25^n}\right)$

Proof:

For $n=1$, the statement reduces to $\dfrac{1}{5^2}=\dfrac{1}{24}\left(1-\dfrac{1}{25}\right)$. This is equivalent to $\dfrac{1}{25}=\dfrac{1}{24}\cdot\dfrac{24}{25}$ and is obviously true.

Assuming the statement is true for $n=k$:

$$\frac{1}{5^2}+\frac{1}{5^4}+\cdots+\frac{1}{5^{2k}}=\frac{1}{24}\left(1-\frac{1}{25^k}\right), \tag{21}$$

we will prove that the statement must be true for $n=k+1$:

$$\frac{1}{5^2}+\frac{1}{5^4}+\cdots+\frac{1}{5^{2(k+1)}}=\frac{1}{24}\left(1-\frac{1}{25^{k+1}}\right). \tag{22}$$

The left-hand side of (22) can be written as

$$\frac{1}{5^2}+\frac{1}{5^4}+\cdots+\frac{1}{5^{2k}}+\frac{1}{5^{2(k+1)}}.$$

In view of (21), this simplifies to:

$$\begin{aligned}\left[\frac{1}{5^2}+\frac{1}{5^4}+\cdots+\frac{1}{5^{2k}}\right]+\frac{1}{5^{2(k+1)}} &= \frac{1}{24}\left(1-\frac{1}{25^k}\right)+\frac{1}{5^{2(k+1)}}\\ &= \frac{25^k-1}{24\cdot 25^k}+\frac{1}{25^{k+1}}\\ &= \frac{25(25^k-1)+24}{24\cdot 25^{k+1}}\\ &= \frac{25^{k+1}-25+24}{24\cdot 25^{k+1}}\\ &= \frac{1}{24}\cdot\frac{25^{k+1}-1}{25^{k+1}}\\ &= \frac{1}{24}\left(1-\frac{1}{25^{k+1}}\right).\end{aligned}$$

Thus the left-hand side of (22) is equal to the right-hand side of (22). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

12. $2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-n} = 1 - 2^{-n}$

Proof:

For $n = 1$, the statement reduces to $2^{-1} = 1 - 2^{-1}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-k} = 1 - 2^{-k}\,, \tag{23}$$

we will prove that the statement must be true for $n = k + 1$:

$$2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-(k+1)} = 1 - 2^{-(k+1)}\,. \tag{24}$$

The left-hand side of (24) can be written as

$$2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-k} + 2^{-(k+1)}.$$

In view of (23), this simplifies to:

$$\begin{aligned}\left(2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-k}\right) + 2^{-(k+1)} &= 1 - 2^{-k} + 2^{-(k+1)} \\ &= 1 - 2 \cdot 2^{-(k+1)} + 2^{-(k+1)} \\ &= 1 - 2^{-(k+1)}.\end{aligned}$$

Thus the left-hand side of (24) is equal to the right-hand side of (24). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

13. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

Proof:

For $n = 1$, the statement reduces to $1 \cdot 2 = \dfrac{1 \cdot 2 \cdot 3}{3}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}\,, \tag{25}$$

we will prove that the statement must be true for $n = k + 1$:

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}\,. \tag{26}$$

The left-hand side of (26) can be written as

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1) + (k+1)(k+2).$$

In view of (25), this simplifies to:

$$\begin{aligned}
[1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k(k+1)] + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \\
&= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} \\
&= \frac{(k+1)(k+2)(k+3)}{3}.
\end{aligned}$$

Thus the left-hand side of (26) is equal to the right-hand side of (26). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

14. $1(2^{-1}) + 2(2^{-2}) + 3(2^{-3}) + \cdots + n(2^{-n}) = 2 - (n+2)2^{-n}$

Proof:

For $n = 1$, the statement reduces to $1(2^{-1}) = 2 - 3 \cdot 2^{-1}$. This is equivalent to $\frac{1}{2} = 2 - \frac{3}{2}$ and is obviously true.

Assuming the statement is true for $n = k$:

$$1(2^{-1}) + 2(2^{-2}) + 3(2^{-3}) + \cdots + k(2^{-k}) = 2 - (k+2)2^{-k}, \tag{27}$$

we will prove that the statement must be true for $n = k+1$:

$$1(2^{-1}) + 2(2^{-2}) + 3(2^{-3}) + \cdots + (k+1)[2^{-(k+1)}] = 2 - (k+3)2^{-(k+1)}. \tag{28}$$

The left-hand side of (28) can be written as

$$1(2^{-1}) + 2(2^{-2}) + 3(2^{-3}) + \cdots + k(2^{-k}) + (k+1)[2^{-(k+1)}].$$

In view of (27), this simplifies to:

$$\begin{aligned}
\left[1(2^{-1}) + 2(2^{-2}) + 3(2^{-3}) + \cdots + k(2^{-k})\right] + (k+1)[2^{-(k+1)}] \\
&= 2 - (k+2)2^{-k} + (k+1)[2^{-(k+1)}] \\
&= 2 - 2(k+2)[2^{-(k+1)}] + (k+1)[2^{-(k+1)}] \\
&= 2 - [2(k+2) - (k+1)][2^{-(k+1)}] \\
&= 2 - (k+3)2^{-(k+1)}.
\end{aligned}$$

Thus the left-hand side of (28) is equal to the right-hand side of (28). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

15. $1(1!) + 2(2!) + 3(3!) + \cdots + n(n!) = (n+1)! - 1$

Proof:

For $n = 1$, the statement reduces to $1(1!) = 2! - 1$ and is obviously true.

Assuming the statement is true for $n = k$:

$$1(1!) + 2(2!) + 3(3!) + \cdots + k(k!) = (k+1)! - 1\,, \tag{29}$$

we will prove that the statement must be true for $n = k + 1$:

$$1(1!) + 2(2!) + 3(3!) + \cdots + (k+1)[(k+1)!] = (k+2)! - 1\,. \tag{30}$$

The left-hand side of (30) can be written as

$$1(1!) + 2(2!) + 3(3!) + \cdots + k(k!) + (k+1)[(k+1)!].$$

In view of (29), this simplifies to:

$$\begin{aligned}
[1(1!) + 2(2!) + 3(3!) + \cdots + k(k!)] &+ (k+1)[(k+1)!] \\
&= (k+1)! - 1 + (k+1)[(k+1)!] \\
&= [1 + (k+1)][(k+1)!] - 1 \\
&= (k+2)[(k+1)!] - 1 \\
&= (k+2)! - 1.
\end{aligned}$$

Thus the left-hand side of (30) is equal to the right-hand side of (30). This proves the inductive step. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

16. Find and verify a formula for the sum

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}\,, \qquad n \geq 1\,.$$

Solution: We can evaluate the sum for the first few values of n:

$$\begin{aligned}
n = 1: &\quad \frac{1}{1\cdot 2} &&= \frac{1}{2} \\
n = 2: &\quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} &&= \frac{2}{3} \\
n = 3: &\quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} &&= \frac{3}{4} \\
n = 4: &\quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} &&= \frac{4}{5} \\
n = 5: &\quad \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} &&= \frac{5}{6}
\end{aligned}$$

The pattern seems to suggest the following formula:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1} \tag{31}$$

We have verified that the formula is valid for $n = 1, 2, 3, 4, 5$. Now we will use mathematical induction to prove that the formula (31) is valid for every positive integer n.

Since the case $n = 1$ has been verified, we will proceed with the inductive step. Assuming the formula (31) is valid for $n = k$:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}, \tag{32}$$

we will prove that (31) must be true for $n = k + 1$:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}. \tag{33}$$

The left-hand side of (33) can be written as

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}.$$

In view of (32), this simplifies to:

$$\begin{aligned}\left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)}\right) &+ \frac{1}{(k+1)(k+2)}\\ &= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}\\ &= \frac{k(k+2)+1}{(k+1)(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{(k+1)^2}{(k+1)(k+2)}\\ &= \frac{k+1}{k+2}.\end{aligned}$$

Thus the left-hand side of (33) is equal to the right-hand side of (33). This proves the inductive step. Therefore, by the principle of mathematical induction, the formula (31) is valid for every positive integer n.

17. Find and verify a formula for the sum $1 + 3 + 5 + \cdots + (2n - 1)$ of the first n odd integers.

Solution: We can evaluate the sum for the first few values of n:

$$\begin{aligned}
n = 1: &\quad 1 &= 1\\
n = 2: &\quad 1+3 &= 4\\
n = 3: &\quad 1+3+5 &= 9\\
n = 4: &\quad 1+3+5+7 &= 16\\
n = 5: &\quad 1+3+5+7+9 &= 25
\end{aligned}$$

The pattern seems to suggest the following formula:

$$1+3+5+\cdots+(2n-1)=n^2\,. \tag{34}$$

We have verified that the formula is valid for $n = 1, 2, 3, 4, 5$. Now we will use mathematical induction to prove that the formula (34) is valid for every positive integer n.

Since the case $n = 1$ has been verified, we will proceed with the inductive step. Assuming the formula (34) is valid for $n = k$:

$$1+3+5+\cdots+(2k-1)=k^2\,, \tag{35}$$

we will prove that (34) must be true for $n = k + 1$:

$$1+3+5+\cdots+[2(k+1)-1]=(k+1)^2\,. \tag{36}$$

The left-hand side of (36) can be written as

$$1+3+5+\cdots+(2k-1)+[2(k+1)-1].$$

In view of (35), this simplifies to:

$$\begin{aligned}[1+3+5+\cdots+(2k-1)]+[2(k+1)-1] &= k^2+[2(k+1)-1]\\ &= k^2+2k+1\\ &= (k+1)^2.\end{aligned}$$

Thus the left-hand side of (36) is equal to the right-hand side of (36). This proves the inductive step. Therefore, by the principle of mathematical induction, the formula (31) is valid for every positive integer n.

18. Prove that 7 divides $8^n - 1$ for $n \geq 1$.

Proof:

For $n = 1$, the statement claims that 7 divides $8 - 1$. This is obviously true.

Assuming the statement is true for $n = k$, i.e.,

$$8^k-1 \quad \text{is divisible by} \quad 7\,, \tag{37}$$

we will prove that the statement must be true for $n = k + 1$:

$$8^{k+1}-1 \quad \text{is divisible by} \quad 7. \tag{38}$$

In order to make use of the inductive hypothesis (37), we can transform the expression $8^{k+1} - 1$ as follows:

$$\begin{aligned} 8^{k+1} - 1 &= 8 \cdot 8^k - 1 \\ &= 8(8^k - 1) + 7. \end{aligned}$$

The last expression must be divisible by 7 since, by the inductive hypothesis, $8^k - 1$ is divisible by 7, and obviously, 7 is divisible by 7. Thus (38) holds and the inductive step is proved. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

19. Prove that 15 divides $4^{2n} - 1$ for $n \geq 1$.

Proof:

For $n = 1$, the statement claims that 15 divides $4^2 - 1$. This is obviously true.

Assuming the statement is true for $n = k$, i.e.,

$$4^{2k} - 1 \quad \text{is divisible by} \quad 15\,, \tag{39}$$

we will prove that the statement must be true for $n = k + 1$:

$$4^{2(k+1)} - 1 \quad \text{is divisible by} \quad 15. \tag{40}$$

In order to make use of the inductive hypothesis (39), we can transform the expression $4^{2(k+1)} - 1$ as follows:

$$\begin{aligned} 4^{2(k+1)} - 1 &= 4^{2k+2} - 1 \\ &= 4^2 \cdot 4^{2k} - 1 \\ &= 16(4^{2k} - 1) + 15. \end{aligned}$$

The last expression must be divisible by 15 since, by the inductive hypothesis, $4^{2k} - 1$ is divisible by 15, and obviously, 15 is divisible by 15. Thus (40) holds and the inductive step is proved. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

20. Prove that 4 divides $3(7^{2n} - 1)$ for $n \geq 1$.

Proof:

For $n = 1$, the statement claims that 4 divides $3(7^2 - 1) = 144$. This is obviously true.

Assuming the statement is true for $n = k$, i.e.,

$$3(7^{2k} - 1) \quad \text{is divisible by} \quad 4\,, \tag{41}$$

we will prove that the statement must be true for $n = k + 1$:

$$3\left[7^{2(k+1)} - 1\right] \quad \text{is divisible by} \quad 4. \tag{42}$$

In order to make use of the inductive hypothesis (41), we can transform the expression $3\left[7^{2(k+1)} - 1\right]$ as follows:

$$\begin{aligned} 3\left[7^{2(k+1)} - 1\right] &= 3\left(7^{2k+2} - 1\right) \\ &= 3\left(7^2 \cdot 7^{2k} - 1\right) \\ &= 3\left[49(7^{2k} - 1) + 48\right] \\ &= 147(7^{2k} - 1) + 144. \end{aligned}$$

The last expression must be divisible by 4 since, by the inductive hypothesis, $4^{2k} - 1$ is divisible by 4, and obviously, 144 is divisible by 4. Thus (42) holds and the inductive step is proved. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.

21. Prove that 5 divides $4^{2n} - 1$ for $n \geq 1$.

Proof:

For $n = 1$, the statement claims that 5 divides $4^2 - 1 = 15$. This is obviously true.

Assuming the statement is true for $n = k$, i.e.,

$$4^{2k} - 1 \quad \text{is divisible by} \quad 5\,, \tag{43}$$

we will prove that the statement must be true for $n = k + 1$:

$$4^{2(k+1)} - 1 \quad \text{is divisible by} \quad 5. \tag{44}$$

In order to make use of the inductive hypothesis (43), we can transform the expression $4^{2(k+1)} - 1$ as follows:

$$\begin{aligned} 4^{2(k+1)} - 1 &= 4^{2k+2} - 1 \\ &= 4^2 \cdot 4^{2k} - 1 \\ &= 16(4^{2k} - 1) + 15. \end{aligned}$$

The last expression must be divisible by 5 since, by the inductive hypothesis, $4^{2k} - 1$ is divisible by 5, and obviously, 15 is divisible by 5. Thus (44) holds and the inductive step is proved. Therefore, by the principle of mathematical induction, the given statement is true for every positive integer n.