

22. Base $n=1$: $x^n - y^n = x - y$ - divisible by $x-y$ - true

Assume that for $n=k$ we have proved that

$x-y$ divides $x^k - y^k$.

We need to prove that ($n=k+1$)

$x-y$ divides $x^{k+1} - y^{k+1}$.

$$\text{Indeed, } x^{k+1} - y^{k+1} = x \cdot x^k - y \cdot y^k$$

$$= x(x^k - y^k) + xy^k - y \cdot y^k$$

$$= x \underbrace{(x^k - y^k)}_{\substack{\text{divisible by} \\ x-y \text{ by assumption}}} + (x-y) \underbrace{y^k}_{\text{divisible by } x-y}$$

$$\text{So, } x^{k+1} - y^{k+1} \text{ is divisible by } x-y, \text{ step is completed.}$$

By PMI, we proved the statement for all $n \geq 1$.

23. Base $n=1$: $5^{2+2} - 24 - 25 = 625 - 24 - 25 = 576$ -

divisible by 576.

Assume that 576 divides $5^{2k+2} - 24k - 25$. We

need to prove that 576 divides $5^{2k+4} - 24(k+1) - 25$.

$$= 25 \cdot 5^{2k+2} - 24k - \cancel{49} = 25(5^{2k+2} - 24k - 25) + 25 \cdot 24k$$

$$+ 25 \cdot 25 - 24k - \cancel{49} = 25 \underbrace{(5^{2k+2} - 24k - 25)}_{\substack{\text{divisible by 576} \\ \text{by assumption}}} + \underbrace{576k + 576}_{\text{clearly divisible by 576}}$$

This completes the step of induction. By PMI, we proved the statement for all $n \geq 1$.

By Ex. 1, 3, if $r \neq 1$, then

$$a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(r^k - 1)}{r - 1}.$$

We choose a to be the first term, r — the quotient of the terms, k so that ar^{k-1} is the last term of summation.

For #2: $a=3, r=3, k=n$.

For #8: $a=2, r=2^2=4, k=n$.

For #11: $a=\frac{1}{5^2}=\frac{1}{25}, r=\frac{1}{5^2}=\frac{1}{25}, k=n$.

For #12: $a=\frac{1}{2}, r=\frac{1}{2}, k=n$.

25. Base $n=0$: $x^{2n+1} + y^{2n+1} = x+y$ — divisible by $x+y$.

Assume that $x+y$ divides $x^{2k+1} + y^{2k+1}$, $k \geq 0$.

We need to prove that $x+y$ divides $x^{2k+3} + y^{2k+3}$.

$$\begin{aligned} \text{Indeed, } x^{2k+3} + y^{2k+3} &= x^2 \cdot x^{2k+1} + y^2 \cdot y^{2k+1} \\ &= x^2(x^{2k+1} + y^{2k+1}) - x^2y^{2k+1} + y^2 \cdot y^{2k+1} \\ &= \underbrace{x^2(x^{2k+1} + y^{2k+1})}_{\substack{\text{divisible by} \\ x+y \text{ by assumption}}} - \underbrace{(x-y)(x+y)y^{2k+1}}_{\substack{\text{clearly divisible} \\ \text{by } x+y}}. \end{aligned}$$

So, $x+y$ divides $x^{2k+3} + y^{2k+3}$. By PMI,
this completes the proof.

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26. Clearly, $n^3 + 6n^2 + 2n$ is a positive integer when n is a positive integer, so we only need to prove that 3 divides $n^3 + 6n^2 + 2n$ for all $n \geq 1$.

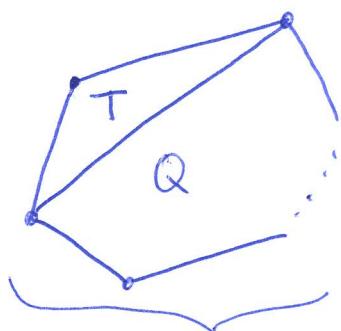
Base: $n=1$: $1^3 + 6 \cdot 1^2 + 2 \cdot 1 = 9$ - divisible by 3.

Assume that 3 divides $k^3 + 6k^2 + 2k$. We need to prove that 3 divides $(k+1)^3 + 6(k+1)^2 + 2(k+1)$

$$\begin{aligned} &= k^3 + 3k^2 + 3k + 1 + 6k^2 + 12k + 6 + 2k + 2 \\ &= k^3 + 9k^2 + 17k + 3 = \underbrace{(k^3 + 6k^2 + 2k)}_{\text{divisible by 3 by assumption}} + \underbrace{3k^2 + 15k + 3}_{\text{clearly divisible by 3 - all coefficients are divisible}} \end{aligned}$$

So, the step is completed, and by PMI the statement is proved for all $n \geq 1$.

27. Any polygon has at least 3 sides. For $n=3$, any polygon with 3 sides is a triangle, and we know that the sum of the interior angles of any triangle is $\pi = (3-2)\pi = (n-2)\pi$. This verifies the base of induction ($n=3$). Assume that we have proved the statement for ~~any~~^{some} $n=k \geq 3$. We need to prove for $n=k+1$.



Let P be a polygon with $k+1$ sides.

We can replace some two consecutive sides by one side to form a ~~new~~ polygon Q with k sides. If T is the triangle formed by the three sides

(two removed from P and one new for Q), then the sum of interior angles of P equals to the sum of interior angles of T (we know it is π) ~~and~~ plus the sum of interior angles of Q , which equals $(k-2)\pi$ by assumption. Adding, we have $\pi + (k-2)\pi = (k-1)\pi = ((k+1)-2)\pi$ as required.

28. If $1+2+\dots+k = \frac{(k+2)(k-1)}{2}$, we need to show

that $1+2+\dots+(k+1) = \frac{(k+3) \cdot k}{2}$. Indeed, by assumption

$$\begin{aligned} \text{LHS} &= (1+2+\dots+k) + (k+1) = \frac{(k+2)(k-1)}{2} + k+1 \\ &= \frac{k^2 + 2k - k - 2 + 2k + 2}{2} = \frac{k^2 + 3k}{2} = \frac{(k+3)k}{2} = \text{RHS}. \end{aligned}$$

The result is NOT valid for all $n \geq 1$ because the base ($n=1$) fails:

$$1 \neq \frac{(1+2)(1-1)}{2} = 0.$$

29. If $1+3+5+\dots+(2k-1) = k^2+4$, we need to prove that $1+3+5+\dots+(2k-1)+(2k+1) = (k+1)^2+4$.

Indeed, by assumption:

$$\text{LHS} = (1+3+\dots+(2k-1)) + (2k+1) = k^2+4+2k+1 = k^2+2k+5$$

$$\text{RHS} = k^2+2k+1+4 = k^2+2k+5.$$

The proposition is not true for all $n \geq 1$, say for $n=1$: LHS = 1, RHS = $1^2+4=5$.

30. Base $n=1$:

$$\begin{array}{ll} \text{LHS} = 1 \\ \text{RHS} = 1^2+1=2 \end{array} \quad \text{true: LHS} < \text{RHS}.$$

Assume for $n=k$ we proved

$$\underbrace{1+3+\dots+(2k-1)}_{\text{LHS}} < k^2+k \quad (\ast)$$

We need to prove for $n=k+1$:

$$\underbrace{1+3+\dots+(2k-1)+(2k+1)}_{\text{LHS}} < \underbrace{(k+1)^2+(k+1)}_{\text{RHS}} \quad (\ast\ast).$$

In order to obtain $(\ast\ast)$, we need to add (\ast) and the following inequality:

$$\underbrace{(2k+1)}_{\text{LHS}} \leq \underbrace{(k+1)^2+(k+1)}_{\text{RHS}} - \underbrace{k^2-k}_{\text{difference}}. \quad (\ast\ast\ast).$$

In other words, $(\ast\ast\ast)$ plus (\ast) gives $(\ast\ast)$.

So, we need to check that $(\ast\ast\ast)$ is true.

$$\text{RHS} = k^2+2k+1+k+1-k^2-k = 2k+2$$

$2k+1 \leq 2k+2$ — true. This completes the step and proof.

31. Base $n=1$: LHS = $2 < \text{RHS} = 1^2 + 2 \cdot 1 = 3$ - true.

Assume that for $n=k$ we proved

$$2 + 4 + \dots + (2k) < k^2 + 2k. \quad (*)$$

We need to prove for $n=k+1$:

$$2 + 4 + \dots + (2k) + (2k+2) < (k+1)^2 + 2(k+1) \quad (**).$$

It is sufficient to show that

$$2k+2 < (k+1)^2 + 2(k+1) - k^2 - 2k \quad (***)$$

because the inequality $(**)$ is the sum of the inequalities $(*)$ and $(***)$.

RHS of $(***)$: $k^2 + 2k + 1 + 2k + 2 - k^2 - 2k = 2k + 3$

So $(***)$ is $2k+2 < 2k+3$ - true.

This completes the step and the proof.

32. Base $n=5$: LHS = $9 \cdot 5! = 9 \cdot 125 = 1125$

$$\text{RHS} = 2^{2 \cdot 5} = 2^{10} = 1024.$$

LHS > RHS - true.

Assume $g(k!) > 2^{2k}$ for $k \geq 5$. We need

to show that $g \cdot ((k+1)!) > 2^{2k+2} = 4 \cdot 2^{2k}$.

Indeed, $g(k+1)! = \underbrace{g \cdot k! \cdot (k+1)}_{\substack{\text{by} \\ \text{assumption}}} > 2^{2k} \cdot (k+1) \geq 4 \cdot 2^{2k}$ because $k+1 \geq 6 > 4$.

This completes the step and the proof.

33. Base $n=2$:

$$\text{LHS} = (1+a)^2 = 1 + 2a + a^2$$

$$\text{RHS} = 1+2a$$

$\text{LHS} > \text{RHS}$ because $a^2 > 0$ (we have $a > 0$).

Assume that we proved for some $n=k \geq 2$:

$$(1+a)^k \geq 1+ka. \quad (*)$$

We need for $n=k+1$:

$$(1+a)^{k+1} \geq 1+(k+1)a \quad (**).$$

Since both sides of both $(*)$ and $(**)$ are positive, it is enough to prove that

$$(1+a) \geq \frac{1+(k+1)a}{1+ka} \quad (***) ,$$

because $(***)$ times $(*)$ gives $(**)$.

$(***)$ is equivalent to:

$$(1+a)(1+ka) \geq 1+(k+1)a$$

$$1+a+ka+ka^2 \geq 1+ka+a$$

- obviously true because $ka^2 > 0$.

This proves $(***)$, and hence $(**)$ and the step of induction.

34 (a) $n=1$ (base): $\text{LHS} = (1+\frac{1}{1}) = 2 \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{true}$
 $\text{RHS} = 1+1=2$

Assume that we proved for $n=k$, i.e.:

$$\left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{k}\right) = k+1.$$

We need to show $n=k+1$:

$$\left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \cdots \left(1 + \frac{1}{k}\right) \left(1 + \frac{1}{k+1}\right) = k+2.$$

Using assumption, we have:

$$\text{LHS} = (k+1) \left(1 + \frac{1}{k+1}\right) = (k+1) \frac{k+1+1}{k+1} = k+2 = \text{RHS}.$$

This proves $n=k+1$, induction step, so by PMI the identity is true for all $n \geq 1$.

(b) Base $n=2$:

$$\begin{aligned} \text{LHS} : \left(1 - \frac{1}{4}\right) &= \frac{3}{4} \\ \text{RHS} : \frac{2+1}{2 \cdot 2} &= \frac{3}{4} \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{true.}$$

Assume $n=k$:

$$\left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}.$$

We need to prove that for $n=k+1$ also true, i.e.:

$$\left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) \cdots \left(1 - \frac{1}{k^2}\right) \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+2}{2k+2}$$

$$\begin{aligned} \text{By assumption, LHS} &= \left(\frac{k+1}{2k}\right) \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+1}{2k} \cdot \frac{k^2+2k+1-1}{(k+1)^2} \\ &= \frac{(k+1) k (k+2)}{2k (k+1)^2} = \frac{k+2}{2(k+1)} = \frac{k+2}{2k+2} = \text{RHS}. \end{aligned}$$

This completes the step of induction and the proof.