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THIRD-ORDER KINEMATIC COMPATIBILITY CONDITIONS FOR A VECTOR FIELD ON AN EMBEDDED SURFACE

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1 Introduction

This paper completes the extension to third order of compatibility conditions, in the sense popularized by T. Y. Thomas [9] (see also [10]), which can be used to study propagation of discontinuities of a vector field defined on a surface in euclidean space. H. Cohen and the author considered such conditions for a general tensor defined on a plane in [1]; here the tensor is restricted to be a three-dimensional vector, but the plane is replaced by a curved surface, extending the work of Cohen and Wang [6] and the author [7]. The third-order results of [1] were subsequently used to study acceleration waves in plates [2, 3], and it is hoped that the present conditions may be useful in similar study, particularly on shells, where the wave may alter the shape of the shell, as [6] was used in [4, 5].

This section serves to introduce the topic of the paper and geometrical notation. Section two describes the kinematic notation and results drawn upon. Section three derives some commutation relations needed in subsequent sections. Section four quotes and derives directional derivatives to compare with the partial derivatives derived in section five. The compatibility conditions are derived in section six for trajectories normal to the singular curves, and in the final section this context is extended to trajectories oblique to those normals.

A smooth surface S in euclidean space E^3 is customarily represented by three scalar functions

$$x^i = x^i(\theta^\alpha) \qquad (i = 1, 2, 3; \alpha = 1, 2), \tag{1.1}$$

where the x^i are cartesian co-ordinates indicating positions and the θ^α, cartesian co-ordinates in E^2, label the points. The summation convention will be used throughout with the ranges 1, 2, 3, for i and 1, 2, for Greek. A moving surface S_t then can be represented by making the x^i functions of time, $x^i = x^i(\theta^\alpha, t)$. In vector notation, $\vec{x} = \vec{x}(\theta^\alpha, t)$. At each point of the surface and at each time t, a basis is supplied for the tangent space of E^3 by the surface's tangent-space basis $\vec{h}_\alpha \equiv \vec{x}_{,\alpha}$, where we are using a comma to indicate partial differentiation, and $\vec{h}_3$, chosen to be perpendicular to $\vec{h}_1$ and $\vec{h}_2$ and to make $(\vec{h}_1, \vec{h}_2, \vec{h}_3)$ a right-hand system. In these terms, the metric of the surface has covariant components $h_{\alpha\beta} \equiv \vec{h}_\alpha \cdot \vec{h}_\beta$ and contravariant components $h^{\alpha\beta}$ such that $h^{\alpha\gamma} h_{\gamma\beta} = \delta^\alpha_\beta$. The reciprocal basis then is given by $\vec{h}^\alpha \equiv h^{\alpha\gamma}\vec{h}_\gamma$, and $\vec{h}^3 \equiv \vec{h}_3$, and the Christoffel symbols and the second fundamental form of the surface by $\left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} \equiv \vec{h}^\alpha \cdot \vec{h}_{\beta,\gamma}$, $\Omega_{\alpha\beta} \equiv \vec{h}_{\alpha,\beta} \cdot \vec{h}_3$, respectively. These relations imply that

$$\vec{h}_{\alpha,\beta} = \left\{ \begin{matrix} \gamma \\ \alpha\beta \end{matrix} \right\} \vec{h}_\gamma + \Omega_{\alpha\beta}\vec{h}_3, \qquad \vec{h}_{3,\beta} = -\Omega^\alpha_\beta \vec{h}_\alpha. \tag{1.2}$$

The concern of this work is with a vector field $\vec{\phi}$, a three-dimensional function of position on the surface, $\vec{\phi} = \vec{\phi}(\theta^\alpha, t) = \phi^\beta \vec{h}_\beta + \phi^3 \vec{h}_3$, where the (contravariant) tangential and (scalar) perpendicular components of $\vec{\phi}$ are, like $\vec{h}_\beta$, $\vec{h}_3$, functions of position on the surface and time. It is necessary to use the covariant derivatives of the tangential components to express succinctly even the partial derivative of $\vec{\phi}$. From the above relations,

$$\vec{\phi}_{,\beta} = (\phi^\alpha_{|\beta} - \phi^3 \Omega^\alpha_\beta)\vec{h}_\alpha + (\phi^3_{,\beta} + \phi^\alpha \Omega_{\alpha\beta})\vec{h}_3. \tag{1.3}$$

One notes the way in which the perpendicular component influences the tangential components of the derivative and *vice versa*. It is the object of this paper to use the method of singular curves to find compatibility conditions that relate the jumps in the third-order partial or covariant derivatives involving time of the component functions of $\vec{\phi}$ to the jumps in their directional derivatives, those of first and second order having been found in [6] and [7] and those of third order not involving time having been found in [8].

The method of singular curves deals with a function that is singular in the sense that it is smooth except on an oriented smooth curve C_t moving on S_t and that for each t it and all of its derivatives must have limits with respect to θ^α and t at C_t from the side to which C_t is moving (the *positive* side of C_t) and the side from which C_t is moving (the *negative* side of C_t). The difference between these limits is called the jump in the function, denoted by square brackets, for example,

$$[\phi^3_{,\beta}] = (\phi^3_{,\beta})^- - (\phi^3_{,\beta})^+. \tag{1.4}$$

The order of the singularity of the function is the lowest order of partial derivative that has a non-zero jump at C_t, order zero meaning that the function itself has a jump. We shall need a formula for the jump of a product in terms of the factors; this formula is as follows,

$$[AB] = A^+[B] + [A]B^+ + [A][B]. \tag{1.5}$$

For each t, we shall have occasion to require the unit tangent of C_t, $\vec{m} = d\vec{x}/ds$, where s is the arc length along C_t according to its given orientation. We need also, in the tangent plane of S_t, the unit normal vector $\vec{n} = \vec{h}_3 \times \vec{m}$, which is generally not the normal vector of the Frenet equations. Frequently we shall have occasion to use the limits of directional derivatives of a singular function as the point of evaluation approaches C_t and the direction approaches that of $\vec{m}$ or $\vec{n}$; when it matters whether the limit is from the positive or negative side of C_t, which one is meant will be made clear. When either is meant, then no indication will be made, for instance, $d\vec{\phi}/dm$ will mean the limit of the directional derivative of $\vec{\phi}$ in the direction $\vec{m}$ at C_t from whichever side is required. In particular, Hadamard's lemma states that

$$\left[\frac{d\vec{\phi}}{dm}\right] = \frac{d\left[\vec{\phi}\right]}{dm}. \tag{1.6}$$

The general assumptions of this paper, following [6], [7], and [8], are that the position vector $\vec{x}$ is singular of order two at C_t, meaning that $\vec{h}_i$, $\vec{h}^i$, $h^{\alpha\beta}$, and $h_{\alpha\beta}$ are continuous, but that the Christoffel symbols and $\Omega_{\alpha\beta}$ are not in general continuous across C_t, and that $\vec{\phi}$ is smooth except at C_t. In addition we assume of $\vec{\phi}$ that its limit at C_t from each side vanishes, that is, $\vec{\phi}^- = 0$ and $\vec{\phi}^+ = 0$, the latter because of its vanishing identically on the positive side of C_t. In consequence $\left[\vec{\phi}\right] = 0$ and all derivatives of $\vec{\phi}$ on the positive side of C_t have vanishing limits at C_t. This is a simplifying assumption of the present work as compared with [7] and [8]. Since the aim of this work is to find formulas for jumps, terms that will ultimately vanish will contribute nothing to those formulas. So long as they will not again be differentiated, they can be discarded; the indication that some such terms have been discarded will be the use of the special equal sign, $\doteq$, which first appears in equation (4.4). Because many terms are products of $\vec{\phi}$ or derivatives of $\vec{\phi}$ in directions tangential to the singular surface and geometrical quantities that may themselves have non-zero jumps across C_t, as mentioned in the previous paragraph, it is important to note the precise effect of the simplifying assumption. In equation (1.5), when A is a geometrical coefficient and B is $\vec{\phi}$, the jump in the product will vanish, the vanishing jump in $\vec{\phi}$ eliminating the first and third terms and the vanishing of $\vec{\phi}$ ahead of C_t eliminating the second. Then by Hadamard's lemma applied twice, the same thing happens with terms involving tangential directional derivatives of $\vec{\phi}$, *e.g.,* with respect to $\vec{m}$. No such term will therefore figure in the final results nor in calculations after the specialization indicated by $\doteq$ has been made.

In the final section, a further simplification will be made that the jump in the normal derivative of the vector field $\tilde{\vec{\chi}}$, defined in equation (4.2), vanishes. The application of this simplifying assumption is indicated by the use of $\approx$.

Despite the lack of an assumption of continuity of the Christoffel symbols and $\Omega_{\alpha\beta}$ across C_t, Cohen and Wang [6] showed that

$$\left[\left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\}\right] m^\beta = 0 \quad \text{and} \quad [\Omega_{\alpha\beta}]\, m^\beta = 0. \tag{1.7}$$

It is necessary to recapitulate the relations among partial differentiation with respect to time, which will be represented by the dot accent, and partial and covariant derivatives with respect to surface co-ordinates. These have been carefully analyzed by Cohen and Wang [6], whose paper should be consulted for details.

2 Kinematics

Much depends upon the velocity vector field $\vec{v}$ of the surface S_t in euclidean space. The so-called world velocity of S_t is then $\vec{\omega} = \vec{v} + \vec{e}_4$, where $\vec{e}_4$ is the unit tangent vector of the t-co-ordinate curve through each point of space. To represent the directional derivative with respect to $\vec{\omega}$, we shall use the dot accent. It will be useful to clarify the use of dots. For vectors, dots are tied to the symbol; $\dot{\vec{\phi}}_{|\alpha\beta}$ is $\left(\dot{\vec{\phi}}\right)_{|\alpha\beta}$ not $\overrightarrow{\dot{\phi}_{|\alpha\beta}}$. For co-ordinates, dots are tied to the co-ordinate expression as a whole but not including differentiation subscripts; $\dot{\phi}^\gamma$ means $\overrightarrow{\phi^\gamma}$ not the $\vec{h}_\gamma$ component of $\dot{\vec{\phi}}$. And $\dot{\phi}^\gamma_{|\lambda}$ means $(\dot{\phi}^\gamma)_{|\lambda}$ not $\overrightarrow{\phi^\gamma_{|\lambda}}$ and not $(\dot{\phi})^\gamma_{|\lambda}$, which has no meaning at all. For any tensor field on S_t, we write

$$\dot{\vec{\phi}} = \frac{\partial \vec{\phi}(\theta^\alpha, t)}{\partial t} = \frac{d\vec{\phi}}{d\omega}. \tag{2.1}$$

In particular, for $\vec{x}(\theta^\alpha, t)$ representing position on S_t, the velocity $\vec{v}(\theta^\alpha, t)$ of S_t is

$$\vec{v} = \frac{d\vec{x}}{d\omega} = \frac{\partial \vec{x}(\theta^\alpha, t)}{\partial t} = \dot{\vec{x}}. \tag{2.2}$$

This velocity can be represented relative to the basis of the tangent space, $\vec{v} = v^\lambda \vec{h}_\lambda + v^3 \vec{h}_3$. It is not assumed continuous across C_t, but it is assumed to be smooth elsewhere. If $\vec{L}$ represents the surface velocity gradient, $\vec{v}_{,\alpha} \otimes \vec{h}^\alpha$, then from (1.2) we have

$$\begin{aligned} \vec{L} &= L^\lambda{}_\alpha \vec{h}_\lambda \otimes \vec{h}^\alpha + L^3{}_\alpha \vec{h}_3 \otimes \vec{h}^\alpha \\ &= (v^\lambda_{|\alpha} - \Omega^\lambda_\alpha v^3)\vec{h}_\lambda \otimes \vec{h}^\alpha + (v^3_{,\alpha} + \Omega_{\lambda\alpha} v^\lambda)\vec{h}_3 \otimes \vec{h}^\alpha. \end{aligned} \tag{2.3}$$

We shall have occasion to expand both $L^\alpha{}_\beta$ and $L^3{}_\beta$ in terms of $\vec{m}$ and $\vec{n}$

$$L^\alpha{}_\beta = L_{mm} m^\alpha m_\beta + L_{mn} m^\alpha n_\beta + L_{nm} n_\alpha m_\beta + L_{nn} n^\alpha n_\beta \tag{2.4}$$

and

$$L^3{}_\beta = L^3{}_m m_\beta + L^3{}_n n_\beta. \tag{2.5}$$

From

$$\dot{\vec{h}}_\alpha = \frac{\partial^2 \vec{x}}{\partial t\, \partial \theta^\alpha} = \frac{\partial^2 \vec{x}}{\partial \theta^\alpha\, \partial t} = \vec{v}_{,\alpha}, \tag{2.6}$$

we have

$$\dot{\vec{h}}_\alpha = L^\lambda{}_\alpha \vec{h}_\lambda + L^3{}_\alpha \vec{h}_3 = (v^\lambda_{|\alpha} - \Omega^\lambda_\alpha v^3)\vec{h}_\lambda + (v^3_{,\alpha} + \Omega_{\lambda\alpha} v^\lambda)\vec{h}_3. \tag{2.7}$$

Then by the perpendicularity of $\vec{h}_\alpha$, $\vec{h}_3$,

$$\dot{\vec{h}}_3 = -L^3{}_\alpha \vec{h}^\alpha = -L^{3\alpha}\vec{h}_\alpha = -(v^{3,\alpha} + \Omega^\alpha_\lambda v^\lambda)\vec{h}_\alpha. \tag{2.8}$$

If the singular curve C_t has an intrinsic speed of propagation U normal to itself across the surface S_t, then the world velocity of C_t can be called $\vec{z} = \vec{\omega} + U\vec{n}$. This relationship among the vectors allows (see [6]) the directional derivative with respect to $\vec{\omega}$ to be written

$$\dot{\vec{\phi}} = \frac{d\vec{\phi}}{dz} - U\frac{d\vec{\phi}}{dn}. \tag{2.9}$$

While the convected displacement derivatives with respect to z have their uses, it has been clearly established by the uses made in [2] and [3] of the derivatives with respect to oblique trajectories used in [1], that compatibility conditions should be found with respect to such oblique trajectories although it is more trouble. Accordingly, let there be added to $\vec{z}$ a tangential velocity, called $\gamma\vec{m}$ in [6], to give the tangential velocity $\vec{\tau} = \vec{z} + \gamma\vec{m}$. Then

$$\frac{d}{d\tau} = \frac{d}{dz} + \gamma\frac{d}{dm}. \tag{2.10}$$

The normal-trajectory derivatives can of course be recovered from the more general expressions simply by letting γ vanish. Finally, since both $\vec{z}$ and $\vec{\tau}$ are tangent to the singular surface swept out in space-time by C_t, there is a version of Hadamard's lemma for directional derivatives with respect to each,

$$\left[\frac{d\vec{\phi}}{dz}\right] = \frac{d[\vec{\phi}]}{dz} = \frac{d\vec{\psi}}{dz}, \qquad \left[\frac{d\vec{\phi}}{d\tau}\right] = \frac{d[\vec{\phi}]}{d\tau} = \frac{d\vec{\psi}}{d\tau}. \tag{2.11}$$

When $\vec{\phi}$ in (2.9) is $\vec{h}_\alpha$, (1.2) produces

$$\frac{d\vec{h}_\alpha}{dz} = \dot{\vec{h}}_\alpha + U\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix} n^\beta \vec{h}_\lambda + U\Omega_{\alpha\beta}n^\beta\vec{h}_3. \tag{2.12}$$

With (2.7) this is

$$\frac{d\vec{h}_\alpha}{dz} = \left(L^\lambda{}_\alpha + U\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix} n^\beta\right)\vec{h}_\lambda + (L^3{}_\alpha + U\Omega_{\lambda\alpha}n^\lambda)\vec{h}_3. \tag{2.13}$$

Then by the perpendicularity of $\vec{h}_\alpha$, $\vec{h}_3$,

$$\frac{d\vec{h}_3}{dz} = -(L^3{}_\alpha + U\Omega_{\alpha\beta}n^\beta)\vec{h}^\alpha = -(L^{3\alpha} + U\Omega^\alpha_\beta n^\beta)\vec{h}_\alpha. \tag{2.14}$$

Both of these formulas are needed later. The continuity of $\vec{h}_\alpha$, $\vec{h}_3$ at C_t together with (2.11), implies

$$\left[L^\lambda{}_\alpha + U\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix} n^\beta\right] = 0, \quad [L^3{}_\alpha + U\Omega_{\alpha\beta}n^\beta] = 0. \tag{2.15}$$

The former of these conditions and (1.7) have two consequences. One is that,

$$\left[L^\lambda{}_\alpha\right] m^\alpha = -U\left[\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix}\right] m^\alpha n^\beta = 0, \tag{2.16}$$

which will be needed later as $[L_{mm}] = [L_{nm}] = 0$, and the other is that, when the operator on co-ordinates,

$$\frac{D\phi^\alpha}{dz} = \frac{d\phi^\alpha}{dz} + \phi^\lambda\left(L^\lambda{}_\alpha + U\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix} n^\beta\right) \tag{2.17}$$

is defined, and when the analogous operator

$$\frac{D\phi^\alpha}{d\tau} = \frac{d\phi^\alpha}{d\tau} + \phi^\lambda\left(L^\lambda{}_\alpha + U\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix} n^\beta + \gamma\begin{Bmatrix}\lambda\\ \alpha\beta\end{Bmatrix} m^\beta\right) = \frac{D\phi^\alpha}{dz} + \gamma\frac{D\phi^\alpha}{Dm} \tag{2.18}$$

is defined, they satisfy versions of Hadamard's lemma,

$$\left[\frac{D\phi^\alpha}{dz}\right] = \frac{D[\phi^\alpha]}{dz} = \frac{D\psi^\alpha}{dz}, \qquad \left[\frac{D\phi^\alpha}{d\tau}\right] = \frac{D[\phi^\alpha]}{d\tau} = \frac{D\psi^\alpha}{d\tau}. \tag{2.19}$$

Four derivatives derived by Cohen and Wang [6] and miscopied in [7] are needed later:

$$\begin{aligned}
\frac{dm_\lambda}{dz} &= \frac{dU}{dm}n_\lambda - L^\alpha{}_\beta m_\alpha m^\beta m_\lambda + U\left\{\begin{matrix}\alpha\\ \lambda\beta\end{matrix}\right\} m_\alpha n^\beta + (L_{\lambda\beta} + L_{\beta\lambda})m^\beta,\\
\frac{dn_\lambda}{dz} &= -\frac{dU}{dm}m_\lambda + L^\alpha{}_\beta n_\alpha n^\beta n_\lambda + U\left\{\begin{matrix}\alpha\\ \lambda\beta\end{matrix}\right\} n_\alpha n^\beta,\\
\frac{dm^\lambda}{dz} &= \frac{dU}{dm}n^\lambda - L^\alpha{}_\beta m_\alpha m^\beta m^\lambda - U\left\{\begin{matrix}\lambda\\ \alpha\beta\end{matrix}\right\} m^\alpha n^\beta,\\
\frac{dn^\lambda}{dz} &= -\frac{dU}{dm}m^\lambda + L^\alpha{}_\beta n_\alpha n^\beta n^\lambda - U\left\{\begin{matrix}\lambda\\ \alpha\beta\end{matrix}\right\} n^\alpha n^\beta - (L^{\lambda\beta} + L^{\beta\lambda})n_\beta.
\end{aligned} \tag{2.20}$$

Using the expansion of $L^\alpha{}_\beta$, one can write, from [7],

$$\frac{Dm^\beta}{dz} = \left(\frac{dU}{dm} + L_{nm}\right)n^\beta, \quad \frac{Dn^\beta}{dz} = -\left(\frac{dU}{dm} + L_{nm}\right)m^\beta, \tag{2.21}$$

. Two more formulas needed from [7] are

$$\dot{L}^{3\alpha} = \frac{DL^{3\alpha}}{dz} - U\frac{DL^{3\alpha}}{Dn} - L^{3\gamma}L^\alpha{}_\gamma, \tag{2.22}$$

and

$$\dot{L}^3{}_\alpha = \frac{DL^3{}_\alpha}{dz} - U\frac{DL^3{}_\alpha}{Dn} + L^3{}_\gamma L^\gamma{}_\alpha. \tag{2.23}$$

3 Commutation Relations

We recall from [6] the definition

$$\vec{\phi}_{|\alpha\beta} = \vec{\phi}_{,\alpha\beta} - \vec{\phi}_{,\gamma}\left\{\begin{matrix}\gamma\\ \alpha\beta\end{matrix}\right\} = \frac{d^2\vec{\phi}}{dh_\beta\, dh_\alpha} - \frac{d\vec{\phi}}{dh_\gamma}\left\{\begin{matrix}\gamma\\ \alpha\beta\end{matrix}\right\}, \tag{3.1}$$

and from [8] the further definition

$$\begin{aligned}
\vec{\phi}_{|\alpha\beta\delta} = \vec{\phi}_{,\beta\alpha\delta} - \vec{\phi}_{,\gamma}\left\{\begin{matrix}\gamma\\ \beta\alpha\end{matrix}\right\}_{,\delta} - \vec{\phi}_{,\gamma\delta}\left\{\begin{matrix}\gamma\\ \beta\alpha\end{matrix}\right\} - \vec{\phi}_{,\theta}\Omega^\theta_\alpha\Omega_{\beta\delta}\\
- \vec{\phi}_{,\theta}\Omega^\theta_\beta\Omega_{\alpha\delta} - \vec{\phi}_{|\eta\beta}\left\{\begin{matrix}\eta\\ \delta\alpha\end{matrix}\right\} - \vec{\phi}_{|\alpha\theta}\left\{\begin{matrix}\theta\\ \delta\beta\end{matrix}\right\}.
\end{aligned} \tag{3.2}$$

Two commutation relations for directional derivatives for fixed time, which will be needed later, can be deduced from expressions for $\vec{\phi}_{|\alpha\beta\delta}$ obtained by iterating

$$\operatorname{grad}\vec{\phi} = \frac{d\vec{\phi}}{dm}\otimes\vec{m} + \frac{d\vec{\phi}}{dn}\otimes\vec{n}, \tag{3.3}$$

in component form,

$$\frac{d\vec{\phi}}{dh_\alpha} = \frac{d\vec{\phi}}{dm}m_\alpha + \frac{d\vec{\phi}}{dn}n_\alpha. \tag{3.4}$$

A number of terms that arise can be simplified using the expression

$$\Omega_{\alpha\beta} = \Omega_{mm} m_\alpha m_\beta + \Omega_{mn} m_\alpha n_\beta + \Omega_{nm} n_\alpha m_\beta + \Omega_{nn} n_\alpha n_\beta \tag{3.5}$$

for the second fundamental form, where $\Omega_{nm} = \Omega_{mn}$ by its symmetry. The expression obtained (from [8] based on [6]) is not identical to but is equal to

$$\begin{aligned}
\vec{\phi}_{|\alpha\beta\delta} = &\left(\frac{d^3\vec{\phi}}{dm^3} - \frac{d\eta}{dm}\frac{d\vec{\phi}}{dn} - 3\eta\frac{d^2\vec{\phi}}{dm\,dn} - 2\eta^2\frac{d\vec{\phi}}{dm} - 2\Omega_{mm}^2\frac{d\vec{\phi}}{dm} \right. \\
&\left. - 2\Omega_{mn}\Omega_{mm}\frac{d\vec{\phi}}{dn} \right) m_\alpha m_\beta m_\delta \\
&+ \left\{ \frac{d^3\vec{\phi}}{dm^2\,dn} + \frac{d\eta}{dm}\frac{d\vec{\phi}}{dm} + 2\eta\frac{d^2\vec{\phi}}{dm^2} - \eta^2\frac{d\vec{\phi}}{dn} - \eta\left(\frac{d^2\vec{\phi}}{dn^2} - \mu\frac{d\vec{\phi}}{dm}\right) \right. \\
&\left. - 2\Omega_{mm}\Omega_{mn}\frac{d\vec{\phi}}{dm} - (\Omega_{mn}^2 + \Omega_{nn}\Omega_{mm})\frac{d\vec{\phi}}{dn} \right\} (m_\alpha m_\beta n_\delta + m_\alpha n_\beta m_\delta + n_\alpha m_\beta m_\delta) \\
&+ \left\{ \left(\frac{d^3\vec{\phi}}{dm\,dn^2} - \frac{d\mu}{dm}\frac{d\vec{\phi}}{dm} - \mu\frac{d^2\vec{\phi}}{dm^2} \right) + 2\eta\frac{d^2\vec{\phi}}{dm\,dn} + 2\eta^2\frac{d\vec{\phi}}{dm} \right. \\
&\left. - 2\Omega_{mn}^2\frac{d\vec{\phi}}{dm} - 2\Omega_{nn}\Omega_{mn}\frac{d\vec{\phi}}{dn} \right\} (m_\alpha n_\beta n_\delta + n_\alpha m_\beta n_\delta + n_\alpha n_\beta m_\delta) \\
&+ \left\{ \left(\frac{d^3\vec{\phi}}{dn^3} - \frac{d\mu}{dn}\frac{d\vec{\phi}}{dm} - \mu\frac{d^2\vec{\phi}}{dn\,dm} \right) - 2\mu\frac{d^2\vec{\phi}}{dm\,dn} \right. \\
&\left. - 2\mu\eta\frac{d\vec{\phi}}{dm} - 2\Omega_{mn}\Omega_{nn}\frac{d\vec{\phi}}{dm} - 2\Omega_{nn}^2\frac{d\vec{\phi}}{dn} \right\} n_\alpha n_\beta n_\delta,
\end{aligned} \tag{3.6}$$

where η is the geodesic curvature of C_t and μ is the geodesic curvature of the orthogonal trajectories on S_t of the family of curves C_t, both defined in terms of covariant directional derivatives by

$$\frac{Dm^\alpha}{Dm} = m^\alpha_{|\lambda} m^\lambda = \eta n^\alpha, \qquad \frac{Dn^\alpha}{Dn} = n^\alpha_{|\lambda} n^\lambda = \mu m^\alpha. \tag{3.7}$$

Because the quantity $\vec{\phi}_{|\alpha\beta\delta}$ is symmetric in the suffixes α, β, and δ, the coefficients of $m_\alpha n_\beta n_\delta$ and $n_\alpha m_\beta n_\delta$ that actually appear in the computation must be equal to that of $n_\alpha n_\beta m_\delta$, which appears above. That is,

$$\begin{aligned}
&\frac{d}{dn}\left(\frac{d^2\vec{\phi}}{dmdn} + \eta\frac{d\vec{\phi}}{dm}\right) - \mu\left(\frac{d^2\vec{\phi}}{dm^2} - \eta\frac{d\vec{\phi}}{dn}\right) + \mu\left(\frac{d^2\vec{\phi}}{dn^2} - \mu\frac{d\vec{\phi}}{dm}\right) \\
&\quad - (\Omega_{mn}^2 + \Omega_{mm}\Omega_{nn})\frac{d\vec{\phi}}{dm} - 2\Omega_{nn}\Omega_{mn}\frac{d\vec{\phi}}{dn} \\
&= \frac{d^3\vec{\phi}}{dm\,dn^2} - \frac{d\mu}{dm}\frac{d\vec{\phi}}{dm} - \mu\frac{d^2\vec{\phi}}{dm^2} + 2\eta\frac{d^2\vec{\phi}}{dm\,dn} + 2\eta^2\frac{d\vec{\phi}}{dm} \\
&\quad - 2\Omega_{mn}^2\frac{d\vec{\phi}}{dm} - 2\Omega_{nn}\Omega_{mn}\frac{d\vec{\phi}}{dn},
\end{aligned} \tag{3.8}$$

which can be written more usefully as

$$\begin{aligned}
\frac{d}{dn}\left(\frac{d^2\vec{\phi}}{dmdn} + \eta\frac{d\vec{\phi}}{dm}\right) = &\frac{d^3\vec{\phi}}{dmdn^2} + 2\eta\frac{d^2\vec{\phi}}{dmdn} - \mu\frac{d^2\vec{\phi}}{dn^2} \\
&+ \left(2\eta^2 + \mu^2 + K - \frac{d\mu}{dm}\right)\frac{d\vec{\phi}}{dm} - \mu\eta\frac{d\vec{\phi}}{dn},
\end{aligned} \tag{3.9}$$

letting

$$K = \Omega_{mm}\Omega_{nn} - \Omega_{mn}^2 \tag{3.10}$$

be the Gaussian curvature of the surface. Likewise the coefficient of $m_\alpha m_\beta n_\delta$ that actually appears must be equal to that of $m_\alpha n_\beta m_\delta$ and $n_\alpha m_\beta m_\delta$, which appears above;

$$\begin{aligned}
&\frac{d}{dn}\left(\frac{d^2\vec{\phi}}{dm^2} - \eta\frac{d\vec{\phi}}{dn}\right) + 2\mu\left(\frac{d^2\vec{\phi}}{dmdn} + \eta\frac{d\vec{\phi}}{dm}\right) \\
&\quad - 2\left(\Omega_{mm}\Omega_{mn}\frac{d\vec{\phi}}{dm} + \Omega_{mn}^2\frac{d\vec{\phi}}{dn}\right) \\
&= \frac{d^3\vec{\phi}}{dm^2\,dn} + \frac{d\eta}{dm}\frac{d\vec{\phi}}{dm} + 2\eta\frac{d^2\vec{\phi}}{dm^2} - \eta^2\frac{d\vec{\phi}}{dn} - \eta\left(\frac{d^2\vec{\phi}}{dn^2} - \mu\frac{d\vec{\phi}}{dm}\right) \\
&\quad - 2\Omega_{mm}\Omega_{mn}\frac{d\vec{\phi}}{dm} - (\Omega_{mn}^2 + \Omega_{nn}\Omega_{mm})\frac{d\vec{\phi}}{dn},
\end{aligned} \tag{3.11}$$

which can be written

$$\begin{aligned}
\frac{d}{dn}\left(\frac{d^2\vec{\phi}}{dm^2} - \eta\frac{d\vec{\phi}}{dn}\right) &= \frac{d^3\vec{\phi}}{dm^2dn} + 2\eta\frac{d^2\vec{\phi}}{dm^2} - 2\mu\frac{d^2\vec{\phi}}{dmdn} - \eta\frac{d^2\vec{\phi}}{dn^2} \\
&\quad + \left(\frac{d\eta}{dm} - \mu\eta\right)\frac{d\vec{\phi}}{dm} - (\eta^2 + K)\frac{d\vec{\phi}}{dn}.
\end{aligned} \tag{3.12}$$

A pair of commutation relations involving differentiation with respect to time will be needed later. They can be derived from the equality of partial derivatives,

$$\frac{\partial^3\vec{\phi}}{\partial t\,\partial\theta^\beta\,\partial\theta^\alpha} = \frac{\partial^3\vec{\phi}}{\partial\theta^\beta\,\partial t\,\partial\theta^\alpha} \tag{3.13}$$

and an expression derived in [6],

$$\dot{\left\{\begin{matrix}\gamma\\ \alpha\beta\end{matrix}\right\}} = L^{3\gamma}\Omega_{\alpha\beta} + L^{\gamma}{}_{\alpha|\beta} - L^3{}_\alpha\Omega^\gamma_\beta, \tag{3.14}$$

expanded in terms of m_β and n_β:

$$\begin{aligned}
\dot{\left\{\begin{matrix}\gamma\\ \alpha\beta\end{matrix}\right\}} &= \left(\frac{DL^\gamma{}_\alpha}{Dm} + L^{3\gamma}(\Omega_{mm}m_\alpha + \Omega_{nm}n_\alpha) - L^3{}_\alpha(\Omega_{mm}m^\gamma + \Omega_{nm}n^\gamma)\right)m_\beta \\
&\quad + \left(\frac{DL^\gamma{}_\alpha}{Dn} + L^{3\gamma}(\Omega_{mn}m_\alpha + \Omega_{nn}n_\alpha) - L^3{}_\alpha(\Omega_{mn}m^\gamma + \Omega_{nn}n^\gamma)\right)n_\beta.
\end{aligned} \tag{3.15}$$

We shall need the coefficient of m_β also expanded in terms of m_α and n_α:

$$\begin{aligned}
&\Big(\frac{dL_{mm}}{dm}m^\gamma + \frac{dL_{nm}}{dm}n^\gamma + \eta L_{mm}n^\gamma - \eta L_{mn}m^\gamma - \eta L_{nm}m^\gamma + \eta L_{nn}n^\gamma \\
&\quad + L^{3n}\Omega_{mm}n^\gamma - L^{3m}\Omega_{nm}n^\gamma\Big)m_\alpha \\
&+ \Big(\frac{dL_{mn}}{dm}m^\gamma + \frac{dL_{nn}}{dm}n^\gamma + \eta L_{mm}m^\gamma + \eta L_{mn}n^\gamma + \eta L_{nm}n^\gamma - \eta L_{nn}m^\gamma \\
&\quad + L^{3m}\Omega_{nm}m^\gamma - L^{3n}\Omega_{mm}m^\gamma\Big)n_\alpha.
\end{aligned} \tag{3.16}$$

The left side of (3.13) is $\dot{\overline{\vec{\phi}_{,\alpha\beta}}} = \dot{\overline{\vec{\phi}_{|\alpha\beta}}} + \dot{\vec{\phi}}_{,\gamma}\left\{ {\gamma \atop \alpha\beta} \right\} + \vec{\phi}_{,\gamma}\dot{\overline{\left\{ {\gamma \atop \alpha\beta} \right\}}}$ and can be expanded using (3.15) and an expansion for $\dot{\overline{\vec{\phi}_{|\alpha\beta}}}$ available by partial differentiation with respect to t of $\vec{\phi}_{|\alpha\beta}$, used in previous work on geometrical compatibility conditions in [8]. We have (cf. (2.9))

$$\begin{aligned}\dot{\overline{\vec{\phi}_{|\alpha\beta}}} = \left(\frac{d}{dz} - U\frac{d}{dn}\right)\Bigg\{&\left(\frac{d^2\vec{\phi}}{dm^2} - \eta\frac{d\vec{\phi}}{dn}\right) m_\alpha m_\beta \\ &+ \left(\frac{d^2\vec{\phi}}{dm\,dn} + \eta\frac{d\vec{\phi}}{dm}\right)(m_\alpha n_\beta + n_\alpha m_\beta) + \left(\frac{d^2\vec{\phi}}{dn^2} - \mu\frac{d\vec{\phi}}{dm}\right) n_\alpha n_\beta\Bigg\}.\end{aligned} \tag{3.17}$$

This differentiation requires the commutation relations (3.9) and (3.12) and results in the expression:

$$\begin{aligned}\dot{\overline{\vec{\phi}_{|\alpha\beta}}} = \Bigg\{&\frac{d^3\vec{\phi}}{dz\,dm^2} - U\frac{d^3\vec{\phi}}{dm^2\,dn} + 2(L_{mm} - U\eta)\frac{d^2\vec{\phi}}{dm^2} - 2\frac{dU}{dm}\frac{d^2\vec{\phi}}{dm\,dn} + U\eta\frac{d^2\vec{\phi}}{dn^2} - \eta\frac{d^2\vec{\phi}}{dz\,dn} \\ &- \left(U\frac{d\eta}{dm} + U\mu\eta + 2\frac{dU}{dm}\eta\right)\frac{d\vec{\phi}}{dm} + \left(U\eta^2 + UK - \frac{d\eta}{dz} - 2L_{mm}\eta\right)\frac{d\vec{\phi}}{dn}\Bigg\} m_\alpha m_\beta \\ &+ \Bigg\{\frac{d^3\vec{\phi}}{dz\,dm\,dn} - U\frac{d^3\vec{\phi}}{dm\,dn^2} + \left(\frac{dU}{dm} + L_{mn} + L_{nm} + U\mu\right)\frac{d^2\vec{\phi}}{dm^2} + (L_{mm} + L_{nn} - 2U\eta)\frac{d^2\vec{\phi}}{dm\,dn} \\ &- \frac{dU}{dm}\frac{d^2\vec{\phi}}{dn^2} + \eta\frac{d^2\vec{\phi}}{dz\,dm} + \left(\frac{d\eta}{dz} + U\frac{d\mu}{dm} - 2U\eta^2 - UK + L_{mm}\eta + L_{nn}\eta + \frac{dU}{dm}\mu\right)\frac{d\vec{\phi}}{dm} \\ &- \eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d\vec{\phi}}{dn}\Bigg\}(m_\alpha n_\beta + n_\alpha m_\beta) \\ &+ \Bigg\{\frac{d^3\vec{\phi}}{dz\,dn^2} - U\frac{d^3\vec{\phi}}{dn^3} + \left(2\frac{dU}{dm} + 3U\mu + 2L_{mn} + 2L_{nm}\right)\frac{d^2\vec{\phi}}{dm\,dn} + 2L_{nn}\frac{d^2\vec{\phi}}{dn^2} - \mu\frac{d^2\vec{\phi}}{dz dm} \\ &+ \left(2\frac{dU}{dm}\eta - \frac{d\mu}{dz} + U\frac{d\mu}{dn} + 3U\mu\eta + 2L_{mn}\eta + 2L_{nm}\eta - 2L_{nn}\mu\right)\frac{d\vec{\phi}}{dm} - U\mu^2\frac{d\vec{\phi}}{dn}\Bigg\} n_\alpha n_\beta.\end{aligned} \tag{3.18}$$

The right side of (3.13) can be obtained by partial differentiation of

$$\begin{aligned}\dot{\vec{\phi}}_{,\alpha} = m_\alpha\Bigg\{&\frac{d^2\vec{\phi}}{dz\,dm} - U\frac{d^2\vec{\phi}}{dm\,dn} + (L_{mm} - U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\Bigg\} \\ &+ n_\alpha\Bigg\{\frac{d^2\vec{\phi}}{dz\,dn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\Bigg\},\end{aligned} \tag{3.19}$$

from [7], where it is equation (4.2_2).

$$
\begin{aligned}
\frac{d}{dm}\dot{\vec{\phi}}_{,\alpha}m_\beta + \frac{d}{dn}\dot{\vec{\phi}}_{,\alpha}n_\beta =& \frac{d}{dm}\left\{\frac{d^2\vec{\phi}}{dz\,dm} - U\frac{d^2\vec{\phi}}{dm\,dn} + (L_{mm}-U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\}m_\alpha m_\beta \\
&- \eta\left\{\frac{d^2\vec{\phi}}{dz\,dn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\}m_\alpha m_\beta \\
&+ \frac{d}{dm}\left\{\frac{d^2\vec{\phi}}{dz\,dn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\}n_\alpha m_\beta \\
&+ \eta\left\{\frac{d^2\vec{\phi}}{dz\,dm} - U\frac{d^2\vec{\phi}}{dm\,dn} + (L_{mm}-U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\}n_\alpha m_\beta \\
&+ \frac{d}{dn}\left\{\frac{d^2\vec{\phi}}{dz\,dm} - U\frac{d^2\vec{\phi}}{dm\,dn} + (L_{mm}-U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\}m_\alpha n_\beta \\
&+ \mu\left\{\frac{d^2\vec{\phi}}{dz\,dn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\}m_\alpha n_\beta \\
&+ \frac{d}{dn}\left\{\frac{d^2\vec{\phi}}{dz\,dn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\}n_\alpha n_\beta \\
&- \mu\left\{\frac{d^2\vec{\phi}}{dz\,dm} - U\frac{d^2\vec{\phi}}{dm\,dn} + (L_{mm}-U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\}n_\alpha n_\beta \\
&+ \dot{\vec{\phi}}_{,\gamma}\begin{Bmatrix}\gamma\\ \alpha\beta\end{Bmatrix},
\end{aligned}
\tag{3.20}
$$

where the partial derivatives of m_α and n_α have been expressed in terms of their covariant derivatives, which gives rise to terms containing Christoffel symbols, collected above into the final term.

When the two above expressions (3.18) and (3.20) for the equal partial derivatives are equated, and the term common to both is dropped, the result as a whole is of no particular interest, but the equality of the coefficients of $n_\alpha m_\beta$ gives one commutation relation needed later and the equality of the coefficients of n_β is the other commutation relation sought. (It could be separated into two relations, but they will be needed together.)

The first equality is

$$
\begin{aligned}
&\frac{d}{dm}\left\{\frac{d^2\vec{\phi}}{dz\,dn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + (L_{nm}+L_{mn}) + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\} \\
&\quad + \eta\left\{\frac{d^2\vec{\phi}}{dz\,dm} - U\frac{d^2\vec{\phi}}{dm\,dn} + (L_{mm}-U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\} \\
&= \frac{d^3\vec{\phi}}{dz\,dm\,dn} - U\frac{d^3\vec{\phi}}{dm\,dn^2} + \left(\frac{dU}{dm} + L_{mn} + L_{nm} + U\mu\right)\frac{d^2\vec{\phi}}{dm^2} + (L_{mm}+L_{nn}-2U\eta)\frac{d^2\vec{\phi}}{dm\,dn} \\
&\quad - \frac{dU}{dm}\frac{d^2\vec{\phi}}{dn^2} + \eta\frac{d^2\vec{\phi}}{dz\,dm} + \left(\frac{d\eta}{dz} + U\frac{d\mu}{dm} - 2U\eta^2 - UK + L_{mm}\eta + L_{nn}\eta + \frac{dU}{dm}\mu\right)\frac{d\vec{\phi}}{dm} \\
&\quad - \eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d\vec{\phi}}{dn} \\
&\quad + \left(\frac{d\vec{\phi}}{dm}m_\gamma + \frac{d\vec{\phi}}{dn}n_\gamma\right)\left\{\left(\frac{dL_{mn}}{dm} + \eta L_{mm} - \eta L_{nn} + L^{3m}\Omega_{nm} - L^{3n}\Omega_{mm}\right)m^\gamma\right. \\
&\qquad \left. + \left(\frac{dL_{nn}}{dm} + \eta L_{mn} + \eta L_{nm}\right)n^\gamma\right\},
\end{aligned}
\tag{3.21}
$$

where (3.16) has been used. It simplifies to

$$\begin{aligned}\frac{d^3\vec{\phi}}{dm\,dz\,dn} &= \frac{d^3\vec{\phi}}{dz\,dm\,dn} + (L_{mm} - U\eta)\frac{d^2\vec{\phi}}{dm\,dn} \\ &\quad - \frac{d}{dm}\left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} - \eta(L_{mm} - U\eta)\frac{d\vec{\phi}}{dm} \\ &\quad + \left(\frac{d\eta}{dz} + U\frac{d\mu}{dm} - 2U\eta^2 - UK + L_{mm}\eta + \frac{dU}{dm}\mu\right)\frac{d\vec{\phi}}{dm} \\ &\quad + \left(\frac{dL_{mn}}{dm} + \eta L_{mm} + L^{3m}\Omega_{nm} - L^{3n}\Omega_{mm}\right)\frac{d\vec{\phi}}{dm} \\ &\doteq \frac{d^3\vec{\phi}}{dz\,dm\,dn} + (L_{mm} - U\eta)\frac{d^2\vec{\phi}}{dm\,dn},\end{aligned} \tag{3.22}$$

The second equality (that of the coefficients of n_β) is

$$\begin{aligned}&\frac{d}{dn}\left\{\frac{d^2\vec{\phi}}{dz\,dm} - U\frac{d^2\vec{\phi}}{dm\,dn} + (L_{mm} - U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\}m_\alpha \\ &\quad + \frac{d}{dn}\left\{\frac{d^2\vec{\phi}}{dz\,dn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\}n_\alpha \\ &= \left\{\frac{d^3\vec{\phi}}{dz\,dm\,dn} - \mu\frac{d^2\vec{\phi}}{dz\,dn} + \eta\frac{d^2\vec{\phi}}{dz\,dm} - U\frac{d^3\vec{\phi}}{dm\,dn^2} + \left(\frac{dU}{dm} + L_{mn} + L_{nm} + U\mu\right)\frac{d^2\vec{\phi}}{dm^2}\right. \\ &\qquad + (L_{mm} + L_{nn} - 2U\eta)\frac{d^2\vec{\phi}}{dm\,dn} + \left(\mu U - \frac{dU}{dm}\right)\frac{d^2\vec{\phi}}{dn^2} \\ &\qquad + \left(\frac{d\eta}{dz} + U\frac{d\mu}{dm} - 2U\eta^2 - U\mu^2 - UK - L_{nm}\mu - L_{mn}\mu + L_{mm}\eta + L_{nn}\eta\right)\frac{d\vec{\phi}}{dm} \\ &\qquad \left. - \left(\frac{dU}{dm}\eta + L_{mn}\eta + L_{nm}\eta + L_{nn}\mu\right)\frac{d\vec{\phi}}{dn}\right\}m_\alpha \\ &\quad + \left\{\frac{d^3\vec{\phi}}{dz\,dn^2} - U\frac{d^3\vec{\phi}}{dn^3} + 2\left(\frac{dU}{dm} + U\mu + L_{mn} + L_{nm}\right)\frac{d^2\vec{\phi}}{dmdn} + 2L_{nn}\frac{d^2\vec{\phi}}{dn^2}\right. \\ &\qquad + \left(2\frac{dU}{dm}\eta - \frac{d\mu}{dz} + U\frac{d\mu}{dn} + 2U\mu\eta + 2L_{mn}\eta + 2L_{nm}\eta - 2L_{nn}\mu + L_{mm}\mu\right)\frac{d\vec{\phi}}{dm} \\ &\qquad \left. - \left(\frac{dU}{dm}\mu + U\mu^2\right)\frac{d\phi}{dn}\right\}n_\alpha \\ &\quad + \left(\frac{d\vec{\phi}}{dm}m_\gamma + \frac{d\vec{\phi}}{dn}n_\gamma\right)\left(\frac{DL^\gamma{}_\alpha}{Dn} + L^{3\gamma}(\Omega_{mn}m_\alpha + \Omega_{nn}n_\alpha) - L^3{}_\alpha(\Omega_{mn}m^\gamma + \Omega_{nn}n^\gamma)\right),\end{aligned} \tag{3.23}$$

where (3.15) has been used.

A commutation relation in co-ordinates that is needed below can be derived by comparison of two equal but different expressions, the n_λ terms in $\dot{\vec{\phi}}_{,\lambda}$ from (4.2) and (4.3) in [7],

$$\frac{d^2\vec{\phi}}{dz\,dn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn} = \frac{d}{dn}\left(\frac{d\vec{\phi}}{dz} - U\frac{d\vec{\phi}}{dn}\right). \tag{3.24}$$

Equating the coefficients of $\vec{h}_\alpha$ and the $\vec{h}_3$ respectively gives

$$\begin{aligned}\frac{D}{dz}\frac{D\phi^\alpha}{Dn} - \phi^\delta\Omega_{\delta\beta}n^\beta(L^{3\alpha} + U\Omega^\alpha_\gamma n^\gamma) = &-\left(U\mu + \frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D\phi^\alpha}{Dm}\\ &-\left(\frac{dU}{dn} + L_{nn}\right)\frac{D\phi^\alpha}{Dn}\\ &+\frac{D^2\phi^\alpha}{Dn\,dz} - \phi^\gamma(L^3_\gamma + U\Omega_{\gamma\lambda}n^\lambda)\Omega^\alpha_\delta n^\delta,\end{aligned}$$

or more usefully

$$\begin{aligned}\frac{D^2\phi^\alpha}{Dn\,dz} = \frac{D^2\phi^\alpha}{dz\,Dn} &+ \left(U\mu + \frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D\phi^\alpha}{Dm} + \left(\frac{dU}{dn} + L_{nn}\right)\frac{D\phi^\alpha}{Dn}\\ &+ \phi^\gamma L^3_\gamma\Omega^\alpha_\delta n^\delta - \phi^\gamma\Omega_{\gamma\delta}n^\delta L^{3\alpha},\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}\frac{d^2\phi^3}{dz\,dn} - \phi^3\Omega^\alpha_\beta n^\beta(L^3{}_\alpha + U\Omega_{\alpha\gamma}n^\gamma) = &-\left(U\mu + \frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d\phi^3}{dm}\\ &-\left(\frac{dU}{dn} + L_{nn}\right)\frac{d\phi^3}{dn}\\ &+\frac{d^2\phi^3}{dn\,dz} - \phi^3(L^{3\alpha} + U\Omega^\alpha_\lambda n^\lambda)\Omega_{\alpha\delta}n^\delta,\end{aligned}$$

or more usefully

$$\frac{d^2\phi^3}{dn\,dz} = \frac{d^2\phi^3}{dz\,dn} + \left(U\mu + \frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d\phi^3}{dm} + \left(\frac{dU}{dn} + L_{nn}\right)\frac{d\phi^3}{dn}. \tag{3.26}$$

4 Directional Derivatives

Since the goal is co-ordinate expressions, it is necessary to introduce co-ordinates for quantities other than $\vec{\phi} = \phi^\alpha\vec{h}_\alpha + \phi^3\vec{h}_3$, $\vec{\phi}_{,\beta}$, $\vec{L}$, $\vec{m}$, $\vec{n}$, and $\vec{v}$. The first directional derivatives are

$$\begin{aligned}\frac{d\vec{\phi}}{dm} &= \left(\frac{D\phi^\alpha}{Dm} - \phi^3\Omega^\alpha_\beta m^\beta\right)\vec{h}_\alpha + \left(\frac{d\phi^3}{dm} + \phi^\alpha\Omega_{\alpha\beta}m^\beta\right)\vec{h}_3,\\ \frac{d\vec{\phi}}{dn} &= \left(\frac{D\phi^\alpha}{Dn} - \phi^3\Omega^\alpha_\beta n^\beta\right)\vec{h}_\alpha + \left(\frac{d\phi^3}{dn} + \phi^\alpha\Omega_{\alpha\beta}n^\beta\right)\vec{h}_3.\end{aligned} \tag{4.1}$$

The jump in the normal directional derivative of $\vec{\phi}$ is defined to be

$$\tilde{\vec{\chi}} = \left[\frac{d\vec{\phi}}{dn}\right] = \left(\left[\frac{D\phi^\alpha}{Dn}\right] - \left[\phi^3\Omega^\alpha_\beta n^\beta\right]\right)\vec{h}_\alpha + \left(\left[\frac{d\phi^3}{dn}\right] + \left[\phi^\alpha\Omega_{\alpha\beta}n^\beta\right]\right)\vec{h}_3. \tag{4.2}$$

If we let

$$\psi^\alpha = [\phi^\alpha], \qquad \psi^3 = [\phi^3], \qquad \tilde{\psi}^\alpha = \left[\frac{D\phi^\alpha}{Dn}\right], \qquad \tilde{\psi}^3 = \left[\frac{d\phi^3}{dn}\right], \tag{4.3}$$

we can note that, while $\tilde{\psi}^\alpha, \tilde{\psi}^3$, are not the co-ordinates of $\tilde{\vec{\chi}}$, hence the use of different letters, they are equal to those co-ordinates on account of the simplifying assumption on $\vec{\phi}$. On account of this simplification, the first-order compatibility conditions (from [6]) become just

$$[\phi^\alpha_{|\beta}] \doteq \tilde{\psi}^\alpha n_\beta, \qquad \left[\dot{\phi}^\alpha\right] \doteq -U\tilde{\psi}^\alpha, \qquad [\phi^3_{,\beta}] \doteq \tilde{\psi}^3 n_\beta, \qquad \left[\dot{\phi}^3\right] \doteq -U\psi^3. \tag{4.4}$$

The second-order directional derivatives are,

$$\begin{aligned}
\frac{d^2\vec{\phi}}{dm^2} &= \left(\frac{D^2\phi^\lambda}{Dm^2} - \phi^\gamma\Omega_{\gamma\beta}\Omega^\lambda_\delta m^\beta m^\delta - 2\frac{d\phi^3}{dm}\Omega^\lambda_\beta m^\beta - \phi^3\frac{D\Omega^\lambda_\beta}{Dm}m^\beta - \eta\phi^3\Omega^\lambda_\beta n^\beta\right)\vec{h}_\lambda \\
&\quad + \left(2\frac{D\phi^\alpha}{Dm}\Omega_{\alpha\beta}m^\beta + \phi^\alpha\frac{D\Omega_{\alpha\beta}}{Dm}m^\beta + \eta\phi^\alpha\Omega_{\alpha\beta}n^\beta + \frac{d^2\phi^3}{dm^2} - \phi^3\Omega^\alpha_\beta\Omega_{\alpha\delta}m^\beta m^\delta\right)\vec{h}_3, \\
\frac{d^2\vec{\phi}}{dn^2} &= \left(\frac{D^2\phi^\lambda}{Dn^2} - \phi^\gamma\Omega_{\gamma\beta}\Omega^\lambda_\delta n^\beta n^\delta - 2\frac{d\phi^3}{dn}\Omega^\lambda_\beta n^\beta - \phi^3\frac{D\Omega^\lambda_\beta}{Dn}n^\beta - \mu\phi^3\Omega^\lambda_\beta m^\beta\right)\vec{h}_\lambda \\
&\quad + \left(2\frac{D\phi^\alpha}{Dn}\Omega_{\alpha\beta}n^\beta + \phi^\alpha\frac{D\Omega_{\alpha\beta}}{Dn}n^\beta + \mu\phi^\alpha\Omega_{\alpha\beta}m^\beta + \frac{d^2\phi^3}{dn^2} - \phi^3\Omega^\alpha_\beta\Omega_{\alpha\delta}n^\beta n^\delta\right)\vec{h}_3, \qquad (4.5) \\
\frac{d^2\vec{\phi}}{dm\,dn} &= \left(\frac{D^2\phi^\lambda}{Dm\,Dn} - \phi^\gamma\Omega_{\gamma\beta}\Omega^\lambda_\delta m^\delta n^\beta - \frac{d\phi^3}{dm}\Omega^\lambda_\beta n^\beta - \frac{d\phi^3}{dn}\Omega^\lambda_\beta m^\beta\right. \\
&\quad \left. - \phi^3\frac{D\Omega^\lambda_\beta}{Dm}n^\beta + \eta\phi^3\Omega^\lambda_\beta m^\beta\right)\vec{h}_\lambda + \left(\frac{D\phi^\alpha}{Dn}\Omega_{\alpha\beta}m^\beta + \frac{D\phi^\alpha}{Dm}\Omega_{\alpha\beta}n^\beta\right. \\
&\quad \left. + \phi^\alpha\frac{D\Omega_{\alpha\beta}}{Dm}n^\beta - \eta\phi^\alpha\Omega_{\alpha\beta}m^\beta + \frac{d^2\phi^3}{dm\,dn} - \phi^3\Omega^\alpha_\eta\Omega_{\alpha\beta}m^\beta n^\eta\right)\vec{h}_3.
\end{aligned}$$

Again let

$$\tilde{\tilde{\vec{\chi}}} = \left[\frac{d^2\vec{\phi}}{dn^2} - \mu\frac{d\vec{\phi}}{dm}\right] = [\vec{\phi}_{|\alpha\beta}n^\alpha n^\beta], \qquad (4.6)$$

and also, in co-ordinates,

$$\tilde{\tilde{\psi}}^\lambda = \left[\frac{D^2\phi^\lambda}{Dn^2} - \mu\frac{D\phi^\lambda}{Dm}\right] \doteq \left[\frac{D^2\phi^\lambda}{Dn^2}\right], \qquad \tilde{\tilde{\psi}}^3 = \left[\frac{d^2\phi^3}{dn^2} - \mu\frac{d\phi^3}{dm}\right] \doteq \left[\frac{d^2\phi^3}{dn^2}\right]. \qquad (4.7)$$

Again, these are not in general the co-ordinates of $\tilde{\tilde{\vec{\chi}}}$;

$$\begin{aligned}
\tilde{\tilde{\vec{\chi}}} &= \left(\tilde{\tilde{\psi}}^\lambda - [\phi^\sigma\Omega_{\sigma\beta}\Omega^\lambda_\gamma n^\beta n^\gamma] - 2\left[\frac{d\phi^3}{dn}\Omega^\lambda_\gamma n^\gamma\right] - \left[\phi^3\frac{D\Omega^\lambda_\gamma}{Dn}n^\gamma\right]\right)\vec{h}_\lambda \\
&\quad + \left(2\left[\frac{D\phi^\alpha}{Dn}\Omega_{\alpha\beta}n^\beta\right] + \left[\phi^\alpha\frac{D\Omega_{\alpha\beta}}{Dn}n^\beta\right] + \tilde{\tilde{\psi}}^3 - [\phi^3\Omega^\alpha_\gamma\Omega_{\alpha\beta}n^\gamma n^\beta]\right)\vec{h}_3.
\end{aligned} \qquad (4.8)$$

Under the present simplifying assumption on $\vec{\phi}$, the second-order compatibility conditions are as follows: first the geometric,

$$\begin{aligned}
[\phi^\lambda_{|\alpha\beta}] &\doteq -\eta\tilde{\psi}^\lambda m_\alpha m_\beta + \frac{D\tilde{\psi}^\lambda}{Dm}(m_\alpha n_\beta + n_\alpha m_\beta) + \tilde{\tilde{\psi}}^\lambda n_\alpha n_\beta, \\
[\phi^3_{|\alpha\beta}] &\doteq -\eta\tilde{\psi}^3 m_\alpha m_\beta + \frac{d\tilde{\psi}^3}{dm}(m_\alpha n_\beta + n_\alpha m_\beta) + \tilde{\tilde{\psi}}^3 n_\alpha n_\beta,
\end{aligned} \qquad (4.9)$$

then the kinematic,

$$
\begin{aligned}
[\dot\phi^\alpha_{|\lambda}] &\doteq \left(-U\frac{D\tilde\psi^\alpha}{Dm}-\frac{dU}{dm}\tilde\psi^\alpha\right)m_\lambda\\
&\quad+\left(\frac{D\tilde\psi^\alpha}{d\tau}-\gamma\frac{D\tilde\psi^\alpha}{Dm}-U\tilde{\tilde\psi}^\alpha-\left[\frac{D\phi^\delta L^\alpha{}_\delta}{Dn}\right]+\left[L_{nn}\frac{D\phi^\alpha}{Dn}\right]\right)n_\lambda,\\
[\dot\phi^3_{,\lambda}] &\doteq \left(-U\frac{d\tilde\psi^3}{dm}-\frac{dU}{dm}\tilde\psi^3\right)m_\lambda\\
&\quad+\left(\frac{d\tilde\psi^3}{d\tau}-\gamma\frac{d\tilde\psi^3}{dm}-U\tilde{\tilde\psi}^3+\left[L_{nn}\frac{d\phi^3}{dn}\right]\right)n_\lambda,\\
[\ddot\phi^\alpha] &\doteq -2U\frac{D\tilde\psi^\alpha}{d\tau}+2U\gamma\frac{D\tilde\psi^\alpha}{Dm}+U^2\tilde{\tilde\psi}^\alpha-\left[\left(\frac{dU}{dz}+UL_{nn}\right)\frac{D\phi^\alpha}{Dn}\right],\\
[\ddot\phi^3] &\doteq -2U\frac{d\tilde\psi^3}{d\tau}+2U\gamma\frac{d\tilde\psi^3}{dm}+U^2\tilde{\tilde\psi}^3-\left[\left(\frac{dU}{dz}+UL_{nn}\right)\frac{d\phi^3}{dn}\right].
\end{aligned}
\tag{4.10}
$$

The formulas (4.10) were obtained by equating expressions known to be equal in the limit at C_t and then taking jumps in those equations. Many terms do not vanish until jumps are taken in them, as can be seen in [7]. It will simplify further work substantially if terms whose jumps will vanish are neglected because when jumps are taken—as they will be—their jumps will vanish even when multiplied by some geometrical quantity whose jump does not vanish. It is therefore as though the above equations had come from the following statements, which are not themselves true but lead to true statements when jumps are taken. They are needed in what follows.

$$
\begin{aligned}
\dot\phi^\alpha_{|\lambda} &\doteq \left(\frac{D^2\phi^\alpha}{d\tau\,Dm}-\gamma\frac{D^2\phi^\alpha}{Dm^2}-U\frac{D^2\psi^\alpha}{Dm\,Dn}-L^\alpha{}_\delta\frac{D\phi^\delta}{Dm}+(L_{mm}-U\eta)\frac{D\phi^\alpha}{Dm}-\frac{dU}{dm}\frac{D\phi^\alpha}{Dn}\right)m_\lambda\\
&\quad+\left(\frac{D^2\psi^\alpha}{d\tau\,Dn}-\gamma\frac{D^2\phi^\alpha}{Dm\,Dn}-U\frac{D^2\phi^\alpha}{Dn^2}-L^\alpha{}_\delta\frac{D\phi^\delta}{Dn}\right.\\
&\qquad\left.+\left\{\frac{dU}{dm}+U\mu+L_{mn}+L_{nm}\right\}\frac{D\phi^\alpha}{Dm}+L_{nn}\frac{D\phi^\alpha}{Dn}\right)n_\lambda,\\
\dot\phi^3_{,\lambda} &\doteq \left(\frac{d^2\phi^3}{d\tau\,dm}-\gamma\frac{d^2\phi^3}{dm^2}-U\frac{d^2\phi^3}{dm\,dn}+(L_{mm}-U\eta)\frac{d\phi^3}{dm}-\frac{dU}{dm}\frac{d\phi^3}{dn}\right)m_\lambda\\
&\quad+\left(\frac{d^2\phi^3}{d\tau\,dn}-\gamma\frac{d^2\phi^3}{dm\,dn}-U\frac{d^2\phi^3}{dn^2}\right.\\
&\qquad\left.+\left\{\frac{dU}{dm}+U\mu+L_{mn}+L_{nm}\right\}\frac{d\phi^3}{dm}+L_{nn}\frac{d\phi^3}{dn}\right)n_\lambda,\\
\ddot\phi^\alpha &\doteq \frac{D^2\phi^\alpha}{d\tau^2}-2\gamma\frac{D^2\phi^\alpha}{d\tau\,Dm}+\gamma^2\frac{D^2\phi^\alpha}{Dm^2}-2U\frac{D^2\phi^\alpha}{d\tau\,Dn}+2U\gamma\frac{D^2\phi^\alpha}{Dm\,Dn}+U^2\frac{D^2\phi^\alpha}{Dn^2}\\
&\quad-\left(\frac{d\gamma}{d\tau}-\gamma U\eta+\gamma L_{mm}+UL_{mn}+UL_{nm}+U^2\mu+U\frac{dU}{dm}\right)\frac{D\phi^\alpha}{Dm}\\
&\quad-\left(\frac{dU}{dz}+UL_{nn}\right)\frac{D\phi^\alpha}{Dn}-2L^\alpha{}_\delta\left(\frac{D\phi^\delta}{d\tau}-U\frac{D\phi^\delta}{Dn}-\gamma\frac{D\phi^\delta}{Dm}\right),\\
\ddot\phi^3 &\doteq \frac{d^2\phi^3}{d\tau^2}-2\gamma\frac{d^2\phi^3}{d\tau\,dm}+\gamma^2\frac{d^2\phi^3}{dm^2}-2U\frac{d^2\phi^3}{d\tau\,dn}+2U\gamma\frac{d^2\phi^3}{dm\,dn}+U^2\frac{d^2\phi^3}{dn^2}\\
&\quad-\left(\frac{d\gamma}{d\tau}-\gamma U\eta+\gamma L_{mm}+UL_{mn}+UL_{nm}+U^2\mu+U\frac{dU}{dm}\right)\frac{d\phi^3}{dm}\\
&\quad-\left(\frac{dU}{dz}+UL_{nn}\right)\frac{d\phi^3}{dn}.
\end{aligned}
\tag{4.11}
$$

Similarly the first-order kinematic compatibility conditions (4.4) are based on larger equations that will be stated in an untrue form needed for substitution later (from [7]), where it is understood that, while no terms have been discarded as they have been above, jumps need to be taken to make the statements true.

$$\begin{aligned}\dot{\phi}^{\alpha} &= \frac{D\phi^{\alpha}}{d\tau} - \gamma\frac{D\phi^{\alpha}}{Dm} - U\frac{D\phi^{\alpha}}{Dn} - L^{\alpha}{}_{\delta}\phi^{\delta},\\ \dot{\phi}^{3} &= \frac{d\phi^{3}}{d\tau} - \gamma\frac{d\phi^{3}}{dm} - U\frac{d\phi^{3}}{dn}.\end{aligned} \tag{4.12}$$

The third-order directional derivatives not involving time are obtained by differentiating the expressions in (4.5):

$$\begin{aligned}\frac{d^3\vec{\phi}}{dm^3} = &\left(\frac{D^3\phi^{\alpha}}{Dm^3} - 3\frac{D\phi^{\delta}}{Dm}\Omega_{\delta\eta}m^{\eta}\Omega^{\alpha}_{\beta}m^{\beta} - 2\phi^{\delta}\frac{D\Omega_{\delta\eta}m^{\eta}}{Dm}\Omega^{\alpha}_{\beta}m^{\beta} - \phi^{\gamma}\Omega_{\gamma\beta}m^{\beta}\frac{D\Omega^{\alpha}_{\delta}m^{\delta}}{Dm}\right.\\ &- 3\frac{d^2\phi^3}{dm^2}\Omega^{\alpha}_{\beta}m^{\beta} - 3\frac{d\phi^3}{dm}\frac{D\Omega^{\alpha}_{\beta}m^{\beta}}{Dm} + \phi^3\Omega^{\gamma}_{\eta}\Omega_{\gamma\delta}m^{\eta}m^{\delta}\Omega^{\alpha}_{\beta}m^{\beta} - \phi^3\frac{D^2\Omega^{\alpha}_{\beta}m^{\beta}}{Dm^2}\Big)\vec{h}_{\alpha}\\ &+\left(3\frac{D^2\phi^{\alpha}}{Dm^2}\Omega_{\alpha\beta}m^{\beta} + 3\frac{D\phi^{\alpha}}{Dm}\frac{D\Omega_{\alpha\beta}m^{\beta}}{Dm} - \phi^{\gamma}\Omega_{\gamma\eta}\Omega^{\alpha}_{\delta}m^{\eta}m^{\delta}\Omega_{\alpha\beta}m^{\beta}\right.\\ &\left.+\phi^{\alpha}\frac{D^2\Omega_{\alpha\beta}m^{\beta}}{Dm^2} + \frac{d^3\phi^3}{dm^3} - 3\frac{d\phi^3}{dm}\Omega^{\alpha}_{\eta}m^{\eta}\Omega_{\alpha\beta}m^{\beta} - 3\phi^3\frac{D\Omega^{\alpha}_{\eta}m^{\eta}}{Dm}\Omega_{\alpha\beta}m^{\beta}\right)\vec{h}_3.\end{aligned} \tag{4.13}$$

$$\begin{aligned}\frac{d^3\vec{\phi}}{dm^2dn} = &\left(\frac{D^3\phi^{\alpha}}{Dm^2Dn} - \frac{D\phi^{\gamma}\Omega_{\gamma\beta}n^{\beta}\Omega^{\alpha}_{\delta}m^{\delta}}{Dm} - \frac{D^2\phi^3\Omega^{\alpha}_{\beta}n^{\beta}}{Dm^2} - 2\frac{d^2\phi^3}{dmdn}\Omega^{\alpha}_{\beta}m^{\beta} - \frac{d\phi^3}{dn}\frac{D\Omega^{\alpha}_{\beta}m^{\beta}}{Dm}\right.\\ &\left.- \frac{D\phi^{\delta}}{Dn}\Omega_{\delta\eta}m^{\eta}\Omega^{\alpha}_{\beta}m^{\beta} - \frac{D\phi^{\delta}\Omega_{\delta\eta}n^{\eta}}{Dm}\Omega^{\alpha}_{\beta}m^{\beta} + \phi^3\Omega^{\delta}_{\eta}n^{\eta}\Omega_{\delta\gamma}m^{\gamma}\Omega^{\alpha}_{\beta}m^{\beta}\right)\vec{h}_{\alpha}\\ &+\left(2\frac{D^2\phi^{\alpha}}{DmDn}\Omega_{\alpha\beta}m^{\beta} + \frac{D\phi^{\alpha}}{Dn}\frac{D\Omega_{\alpha\beta}m^{\beta}}{Dm} + \frac{D^2\phi^{\alpha}\Omega_{\alpha\beta}n^{\beta}}{Dm^2} - \phi^{\gamma}\Omega_{\gamma\eta}n^{\eta}\Omega^{\alpha}_{\delta}m^{\delta}\Omega_{\alpha\beta}m^{\beta}\right.\\ &\left.+\frac{d^3\phi^3}{dm^2dn} - \frac{d\phi^3\Omega^{\alpha}_{\eta}n^{\eta}\Omega_{\alpha\beta}m^{\beta}}{dm} - \frac{d\phi^3\Omega^{\alpha}_{\eta}n^{\eta}}{dm}\Omega_{\alpha\beta}m^{\beta} - \frac{d\phi^3}{dn}\Omega^{\alpha}_{\eta}m^{\eta}\Omega_{\alpha\beta}m^{\beta}\right)\vec{h}_3.\end{aligned} \tag{4.14}$$

$$\begin{aligned}\frac{d^3\vec{\phi}}{dm\,dn^2} = &\left(\frac{D^3\phi^{\alpha}}{Dm\,Dn^2} - \frac{D\phi^{\lambda}}{Dm}\Omega_{\lambda\beta}\Omega^{\alpha}_{\gamma}n^{\beta}n^{\gamma} - \phi^{\lambda}\frac{D\Omega_{\lambda\beta}\Omega^{\alpha}_{\gamma}n^{\beta}n^{\gamma}}{Dm} - 2\frac{d^2\phi^3}{dm\,dn}\Omega^{\alpha}_{\gamma}n^{\gamma}\right.\\ &- 2\frac{d\phi^3}{dn}\frac{D\Omega^{\alpha}_{\gamma}n^{\gamma}}{Dm} - \frac{d\phi^3}{dm}\frac{D\Omega^{\alpha}_{\gamma}n^{\gamma}}{Dn} - \phi^3\frac{D^2\Omega^{\alpha}_{\gamma}n^{\gamma}}{Dm\,Dn} - \frac{d^2\phi^3}{dn^2}\Omega^{\alpha}_{\beta}m^{\beta}\\ &\left.+\phi^3\Omega^{\delta}_{\gamma}\Omega_{\delta\eta}n^{\gamma}n^{\eta}\Omega^{\alpha}_{\beta}m^{\beta} - 2\frac{D\phi^{\delta}}{Dn}\Omega_{\delta\eta}n^{\eta}\Omega^{\alpha}_{\beta}m^{\beta} - \phi^{\delta}\frac{D\Omega_{\delta\eta}n^{\eta}}{Dn}\Omega^{\alpha}_{\beta}m^{\beta}\right)\vec{h}_{\alpha}\\ &+\left(\frac{D^2\phi^{\alpha}}{Dn^2}\Omega_{\alpha\beta}m^{\beta} - \phi^{\lambda}\Omega_{\lambda\eta}\Omega^{\alpha}_{\gamma}n^{\eta}n^{\gamma}\Omega_{\alpha\beta}m^{\beta} - 2\frac{d\phi^3}{dn}\Omega^{\alpha}_{\gamma}n^{\gamma}\Omega_{\alpha\beta}m^{\beta}\right.\\ &- \phi^3\frac{D\Omega^{\alpha}_{\gamma}n^{\gamma}}{Dn}\Omega_{\alpha\beta}m^{\beta} + \frac{d^3\phi^3}{dm\,dn^2} - \frac{d\phi^3}{dm}\Omega^{\alpha}_{\gamma}\Omega_{\alpha\beta}n^{\gamma}n^{\beta} - \phi^3\frac{d\Omega^{\alpha}_{\gamma}\Omega_{\alpha\beta}n^{\gamma}n^{\beta}}{dm}\\ &\left.+2\frac{D^2\phi^{\alpha}}{Dm\,Dn}\Omega_{\alpha\beta}n^{\beta} + 2\frac{D\phi^{\alpha}}{Dn}\frac{D\Omega_{\alpha\beta}n^{\beta}}{Dm} + \frac{D\phi^{\alpha}}{Dm}\frac{D\Omega_{\alpha\beta}n^{\beta}}{Dn} + \phi^{\alpha}\frac{D^2\Omega_{\alpha\beta}n^{\beta}}{Dm\,Dn}\right)\vec{h}_3\end{aligned} \tag{4.15}$$

$$\begin{aligned}\frac{d^3\vec{\phi}}{dn^3} &= \left(\frac{D^3\phi^\alpha}{Dn^3} - \frac{D\phi^\lambda\Omega_{\lambda\beta}\Omega^\alpha_\gamma n^\beta n^\gamma}{Dn} - 2\frac{D\phi^\gamma}{Dn}\Omega_{\gamma\delta}n^\delta\Omega^\alpha_\beta n^\beta - \phi^\gamma\frac{D\Omega_{\gamma\delta}n^\delta}{Dn}\Omega^\alpha_\beta n^\beta\right.\\ &\qquad \left. - 3\frac{d^2\phi^3}{dn^2}\Omega^\alpha_\gamma n^\gamma - 3\frac{d\phi^3}{dn}\frac{D\Omega^\alpha_\gamma n^\gamma}{Dn} - \phi^3\frac{D^2\Omega^\alpha_\gamma n^\gamma}{Dn^2} + \phi^3\Omega^\eta_\gamma\Omega_{\eta\delta}n^\gamma n^\delta\Omega^\alpha_\beta n^\beta\right)\vec{h}_\alpha\\ &+ \left(3\frac{D^2\phi^\alpha}{Dn^2}\Omega_{\alpha\beta}n^\beta + 3\frac{D\phi^\alpha}{Dn}\frac{D\Omega_{\alpha\beta}n^\beta}{Dn} + \phi^\alpha\frac{D^2\Omega_{\alpha\beta}n^\beta}{Dn^2} - \phi^\lambda\Omega_{\lambda\eta}\Omega^\alpha_\gamma n^\eta n^\gamma\Omega_{\alpha\beta}n^\beta\right.\\ &\qquad \left. + \frac{d^3\phi^3}{dn^3} - \frac{d\phi^3\Omega^\alpha_\gamma\Omega_{\alpha\beta}n^\gamma n^\beta}{dn} - 2\frac{d\phi^3}{dn}\Omega^\alpha_\gamma n^\gamma\Omega_{\alpha\beta}n^\beta - \phi^3\frac{D\Omega^\alpha_\gamma n^\gamma}{Dn}\Omega_{\alpha\beta}n^\beta\right)\vec{h}_3.\end{aligned} \tag{4.16}$$

Using the notation $\vec{\tilde{\chi}}$ for $[\vec{\phi}_{|\alpha\beta\delta}n^\alpha n^\beta n^\delta]$, which is available as the coefficient of $n_\alpha n_\beta n_\delta$ in the expression (3.6) for $\phi_{|\alpha\beta\delta}$, we also need

$$\begin{aligned}\tilde{\tilde{\psi}}^\alpha &= \left[\frac{D^3\phi^\alpha}{Dn^3} - \frac{d\mu}{dn}\frac{D\phi^\alpha}{Dm} - 3\mu\frac{D^2\phi^\alpha}{Dm\,Dn} - 3\mu\eta\frac{D\phi^\alpha}{Dm} + \mu^2\frac{D\phi^\alpha}{Dn} + 2\mu\phi^\nu R^\alpha{}_{\nu\gamma\beta}m^\gamma n^\beta\right]\\ &\doteq \left[\frac{D^3\phi^\alpha}{Dn^3} - 3\mu\frac{D^2\phi^\alpha}{Dm\,Dn} + \mu^2\frac{D\phi^\alpha}{Dn}\right]\end{aligned} \tag{4.17}$$

and

$$\tilde{\tilde{\psi}}^3 = \left[\frac{d^3\phi^3}{dn^3} - \frac{d\mu}{dn}\frac{d\phi^3}{dm} - 3\mu\frac{d^2\phi^3}{dm\,dn} - 3\mu\eta\frac{d\phi^3}{dm} + \mu^2\frac{d\phi^3}{dn}\right] \doteq \left[\frac{d^3\phi^3}{dn^3} - 3\mu\frac{d^2\phi^3}{dm\,dn} + \mu^2\frac{d\phi^3}{dn}\right]. \tag{4.18}$$

With the notation needed all in place, the third-order geometric conditions of [8] under the simplifying assumption of this work can now be written out:

$$\begin{aligned}\phi^\lambda_{|\alpha\beta\delta} &\doteq \left(-\frac{d\eta}{dm}\tilde{\psi}^\lambda - 3\eta\frac{D\tilde{\psi}^\lambda}{Dm}\right)m_\alpha m_\beta m_\delta\\ &+ \left(\frac{D^2\tilde{\psi}^\lambda}{Dm^2} - \eta^2\tilde{\psi}^\lambda - \eta\tilde{\tilde{\psi}}^\lambda\right)(m_\alpha m_\beta n_\delta + m_\alpha n_\beta m_\delta + n_\alpha m_\beta m_\delta)\\ &+ \left[\frac{D\phi^\nu}{Dn}R^\lambda{}_{\nu\gamma\sigma}m^\sigma n^\gamma\right](2m_\alpha n_\beta n_\delta + n_\alpha m_\beta n_\delta)\\ &+ \left(\frac{D\tilde{\tilde{\psi}}^\lambda}{Dm} + 2\eta\frac{D\tilde{\psi}^\lambda}{Dm}\right)(m_\alpha n_\beta n_\delta + n_\alpha m_\beta n_\delta + n_\alpha n_\beta m_\delta)\\ &- \left[K\frac{D\phi^\lambda}{Dn}\right]m_\alpha m_\beta n_\delta + \tilde{\tilde{\psi}}^\lambda n_\alpha n_\beta n_\delta\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}\phi^3_{|\alpha\beta\delta} &\doteq \left(-\frac{d\eta}{dm}\tilde{\psi}^3 - 3\eta\frac{d\tilde{\psi}^3}{dm}\right)m_\alpha m_\beta m_\delta\\ &+ \left(\frac{d^2\tilde{\psi}^3}{dm^2} - \eta^2\tilde{\psi}^3 - \eta\tilde{\tilde{\phi}}^3\right)(m_\alpha m_\beta n_\delta + m_\alpha n_\beta m_\delta + n_\alpha m_\beta m_\delta)\\ &+ \left(\frac{d\tilde{\tilde{\psi}}^3}{dm} + 2\eta\frac{d\tilde{\psi}^3}{dm}\right)(m_\alpha n_\beta n_\delta + n_\alpha m_\beta n_\delta + n_\alpha n_\beta m_\delta)\\ &- \left[K\frac{d\phi^3}{dn}\right]m_\alpha m_\beta n_\delta + \tilde{\tilde{\psi}}^3 n_\alpha n_\beta n_\delta.\end{aligned} \tag{4.20}$$

Directional derivatives involving time will be needed. From (4.5), (4.17), and (4.18) in [7] come:

$$\begin{aligned}\frac{d^2\vec{\phi}}{dz\,dm} = &\Big(\frac{D^2\phi^\alpha}{dz\,Dm} - \phi^\gamma\Omega_{\gamma\beta}m^\beta L^{3\alpha} - U\phi^\gamma\Omega_{\gamma\beta}m^\beta\Omega^\alpha_\delta n^\delta\\ &- \frac{d\phi^3}{dz}\Omega^\alpha_\beta m^\beta - \phi^3\frac{D\Omega^\alpha_\beta}{dz}m^\beta - \frac{dU}{dm}\phi^3\Omega^\alpha_\beta n^\beta\\ &- L_{nm}\phi^3\Omega^\alpha_\beta n^\beta - \frac{d\phi^3}{dm}L^{3\alpha} - U\frac{d\phi^3}{dm}\Omega^\alpha_\gamma n^\gamma\Big)\vec{h}_\alpha\\ &+\Big(\frac{d^2\phi^3}{dz\,dm} - \phi^3\Omega^\alpha_\beta m^\beta L^3{}_\alpha - U\phi^3\Omega^\alpha_\beta m^\beta\Omega_{\alpha\gamma}n^\gamma\\ &+ \frac{D\phi^\alpha}{dz}\Omega_{\alpha\beta}m^\beta + \phi^\alpha\frac{D\Omega_{\alpha\beta}}{dz}m^\beta + \frac{dU}{dm}\phi^\alpha\Omega_{\alpha\beta}n^\beta\\ &+ L_{nm}\phi^\alpha\Omega_{\alpha\beta}n^\beta + \frac{D\phi^\alpha}{Dm}L^3{}_\alpha + U\frac{D\phi^\alpha}{Dm}\Omega_{\alpha\delta}n^\delta\Big)\vec{h}_3.\end{aligned} \tag{4.21}$$

$$\begin{aligned}\frac{d^2\vec{\phi}}{dz\,dn} = &\Big(\frac{D^2\phi^\alpha}{dz\,Dn} - \phi^\gamma\Omega_{\gamma\beta}n^\beta L^{3\alpha} - U\phi^\gamma\Omega_{\gamma\beta}n^\beta\Omega^\alpha_\delta n^\delta - \frac{d\phi^3}{dz}\Omega^\alpha_\beta n^\beta - \phi^3\frac{D\Omega^\alpha_\beta}{dz}n^\beta\\ &+ \frac{dU}{dm}\phi^3\Omega^\alpha_\beta m^\beta + L_{nm}\phi^3\Omega^\alpha_\beta m^\beta - \frac{d\phi^3}{dn}L^{3\alpha} - U\frac{d\phi^3}{dn}\Omega^\alpha_\gamma n^\gamma\Big)\vec{h}_\alpha\\ &+\Big(\frac{d^2\phi^3}{dz\,dn} - \phi^3\Omega^\alpha_\beta n^\beta L^3{}_\alpha - U\phi^3\Omega^\alpha_\beta n^\beta\Omega_{\alpha\gamma}n^\gamma + \frac{D\phi^\gamma\Omega_{\gamma\beta}}{dz}n^\beta\\ &- \frac{dU}{dm}\phi^\gamma\Omega_{\gamma\beta}m^\beta - L_{nm}\phi^\gamma\Omega_{\gamma\beta}m^\beta + \frac{D\phi^\alpha}{Dn}L^3{}_\alpha + U\frac{D\phi^\alpha}{Dn}\Omega_{\alpha\gamma}n^\gamma\Big)\vec{h}_3,\end{aligned} \tag{4.22}$$

$$\begin{aligned}\frac{d^2\vec{\phi}}{dz^2} = &\Big\{\frac{D^2\phi^\alpha}{dz^2} - \phi^\gamma L^3{}_\gamma L^{3\alpha} - U\phi^\gamma\Omega_{\gamma\delta}n^\delta L^{3\alpha} - U\phi^\gamma\Omega^\alpha_\lambda n^\lambda L^3{}_\gamma\\ &- U^2\phi^\gamma\Omega_{\gamma\delta}n^\delta\Omega^\alpha_\lambda n^\lambda - 2\frac{d\phi^3}{dz}L^{3\alpha} - 2U\frac{d\phi^3}{dz}\Omega^\alpha_\lambda n^\lambda\\ &- \phi^3\Big(\frac{DL^{3\alpha}}{dz} + \frac{DU\Omega^\alpha_\lambda}{dz}n^\lambda - U\Big(\frac{dU}{dm} + L_{nm}\Big)\Omega^\alpha_\lambda m^\lambda\Big)\Big\}\vec{h}_\alpha\\ &+\Big\{\frac{d^2\phi^3}{dz^2} - \phi^3L^3{}_\gamma L^{3\gamma} - U\phi^3\Omega_{\gamma\delta}n^\delta L^{3\gamma} - U\phi^3\Omega^\gamma_\lambda n^\lambda L^3{}_\gamma\\ &- U^2\phi^3\Omega_{\gamma\delta}n^\delta\Omega^\gamma_\lambda n^\lambda + 2\frac{D\phi^\alpha}{dz}L^3{}_\alpha + 2U\frac{D\phi^\alpha}{dz}\Omega_{\alpha\beta}n^\beta\\ &+ \phi^\alpha\Big(\frac{DL^3{}_\alpha}{dz} + \frac{DU\Omega_{\alpha\lambda}}{dz}n^\lambda - U\Big(\frac{dU}{dm} + L_{nm}\Big)\Omega_{\alpha\lambda}m^\lambda\Big)\Big\}\vec{h}_3.\end{aligned} \tag{4.23}$$

Using the general formula, from (2.13) and (2.14),

$$\begin{aligned}\frac{d(A^\alpha\vec{h}_\alpha + B\vec{h}_3)}{dz} =&\Big\{\frac{DA^\alpha}{dz} - B(L^{3\alpha} + U\Omega^\alpha_\beta n^\beta)\Big\}\vec{h}_\alpha\\ &+\Big\{A^\alpha(L^3{}_\alpha + U\Omega_{\lambda\alpha}n^\lambda) + \frac{dB}{dz}\Big\}\vec{h}_3,\end{aligned} \tag{4.24}$$

further partly temporal directional derivatives can be calculated from (4.5). Because these expressions will

not be differentiated again, terms in ϕ^α and ϕ^3 will be dropped.

$$\begin{aligned}
\frac{d^3\vec{\phi}}{dz\,dm^2} \doteq \Bigg\{ & \frac{D^3\phi^\alpha}{dz\,Dm^2} - \frac{D\phi^\gamma}{dz}\Omega_{\gamma\beta}m^\beta\Omega^\alpha_\delta m^\delta - 2\frac{D\phi^\gamma}{Dm}\Omega_{\gamma\beta}m^\beta(L^{3\alpha}+U\Omega^\alpha_\delta n^\delta) \\
& - 2\frac{d^2\phi^3}{dz\,dm}\Omega^\alpha_\gamma m^\gamma - \frac{d^2\phi^3}{dm^2}(L^{3\alpha}+U\Omega^\alpha_\gamma n^\gamma) - \frac{d\phi^3}{dz}\left(\frac{D\Omega^\alpha_\gamma}{Dm}m^\gamma + \eta\Omega^\alpha_\gamma n^\gamma\right) \\
& - 2\frac{d\phi^3}{dm}\left(\frac{dU}{dm}\Omega^\alpha_\gamma n^\gamma + \frac{D\Omega^\alpha_\gamma}{dz}m^\gamma + L_{nm}\Omega^\alpha_\gamma n^\gamma\right)\Bigg\}\vec{h}_\alpha \\
+ \Bigg\{ & \frac{d^3\phi^3}{dz\,dm^2} - \frac{d\phi^3}{dz}\Omega_{\alpha\theta}\Omega^\theta_\beta m^\alpha m^\beta - 2\frac{d\phi^3}{dm}\Omega_{\alpha\gamma}m^\gamma(L^{3\alpha}+U\Omega^\alpha_\beta n^\beta) \\
& + 2\frac{D^2\phi^\alpha}{dz\,Dm}\Omega_{\alpha\gamma}m^\gamma + \frac{D^2\phi^\alpha}{Dm^2}(L^3{}_\alpha + U\Omega_{\alpha\gamma}n^\gamma) + \frac{D\phi^\alpha}{dz}\left(\frac{D\Omega_{\alpha\gamma}}{Dm}m^\gamma + \eta\Omega_{\alpha\gamma}n^\gamma\right) \\
& + 2\frac{D\phi^\alpha}{Dm}\left(\frac{dU}{dm}\Omega_{\alpha\gamma}n^\gamma + \frac{D\Omega_{\alpha\gamma}}{dz}m^\gamma + L_{nm}\Omega_{\alpha\gamma}n^\gamma\right)\Bigg\}\vec{h}_3.
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
\frac{d^3\vec{\phi}}{dz\,dm\,dn} \doteq \Bigg\{ & \frac{D^3\phi^\alpha}{dz\,Dm\,Dn} - \frac{D\phi^\gamma}{dz}\Omega_{\gamma\delta}n^\delta\Omega^\alpha_\beta m^\beta - \frac{D\phi^\gamma}{Dn}\Omega_{\gamma\beta}m^\beta(L^{3\alpha}+U\Omega^\alpha_\delta n^\delta) \\
& - \frac{D\phi^\gamma}{Dm}\Omega_{\gamma\beta}n^\beta(L^{3\alpha}+U\Omega^\alpha_\delta n^\delta) - \frac{d^2\phi^3}{dz\,dn}\Omega^\alpha_\gamma m^\gamma - \frac{d^2\phi^3}{dz\,dm}\Omega^\alpha_\gamma n^\gamma \\
& + \frac{d\phi^3}{dz}\left(-\frac{D\Omega^\alpha_\gamma}{Dm}n^\gamma + \eta\Omega^\alpha_\gamma m^\gamma\right) - \frac{d^2\phi^3}{dm\,dn}(L^{3\alpha}+U\Omega^\alpha_\gamma n^\gamma) \\
& + \frac{d\phi^3}{dm}\left(\frac{dU}{dm}\Omega^\alpha_\gamma m^\gamma - \frac{D\Omega^\alpha_\gamma}{dz}n^\gamma + L_{nm}\Omega^\alpha_\gamma m^\gamma\right) \\
& - \frac{d\phi^3}{dn}\left(\frac{dU}{dm}\Omega^\alpha_\gamma n^\gamma + \frac{D\Omega^\alpha_\gamma}{dz}m^\gamma + L_{nm}\Omega^\alpha_\gamma n^\gamma\right)\Bigg\}\vec{h}_\alpha \\
+ \Bigg\{ & \frac{d^3\phi^3}{dz\,dm\,dn} - \frac{d\phi^3}{dz}\Omega_{\alpha\theta}\Omega^\theta_\beta m^\alpha n^\beta - \frac{d\phi^3}{dn}\Omega_{\alpha\gamma}m^\gamma(L^{3\alpha}+U\Omega^\alpha_\beta n^\beta) \\
& - \frac{d\phi^3}{dm}\Omega_{\alpha\gamma}n^\gamma(L^{3\alpha}+U\Omega^\alpha_\beta n^\beta) + \frac{D^2\phi^\alpha}{dz\,Dm}\Omega_{\alpha\beta}n^\beta + \frac{D^2\phi^\alpha}{dz\,Dn}\Omega_{\alpha\beta}m^\beta \\
& + \frac{D\phi^\alpha}{dz}\left(\frac{D\Omega_{\alpha\gamma}}{Dm}n^\gamma - \eta\Omega_{\alpha\gamma}m^\gamma\right) - \frac{D^2\phi^\alpha}{Dm\,Dn}(L^3{}_\alpha + U\Omega_{\alpha\gamma}n^\gamma) \\
& - \frac{D\phi^\alpha}{Dm}\left(\frac{dU}{dm}\Omega_{\alpha\gamma}m^\gamma - \frac{D\Omega_{\alpha\gamma}}{dz}n^\gamma + L_{nm}\Omega_{\alpha\gamma}m^\gamma\right) \\
& + \frac{D\phi^\alpha}{Dn}\left(\frac{dU}{dm}\Omega_{\alpha\gamma}n^\gamma + \frac{D\Omega_{\alpha\gamma}}{dz}m^\gamma + L_{nm}\Omega_{\alpha\gamma}n^\gamma\right)\Bigg\}\vec{h}_3.
\end{aligned} \tag{4.26}$$

$$
\begin{aligned}
\frac{d^3\vec{\phi}}{dz\,dn^2} \doteq \Bigg\{ & \frac{D^3\phi^\alpha}{dz\,Dn^2} - \frac{D\phi^\gamma}{dz}\Omega_{\gamma\beta}n^\beta\Omega^\alpha_\delta n^\delta - 2\frac{D\phi^\gamma}{Dn}\Omega_{\gamma\beta}n^\beta(L^{3\alpha} + U\Omega^\alpha_\delta n^\delta) \\
& - 2\frac{d^2\phi^3}{dz\,dn}\Omega^\alpha_\gamma n^\gamma - \frac{d^2\phi^3}{dn^2}(L^{3\alpha} + U\Omega^\alpha_\gamma n^\gamma) - \frac{d\phi^3}{dz}\left(\frac{D\Omega^\alpha_\gamma}{Dn}n^\gamma + \mu\Omega^\alpha_\gamma m^\gamma\right) \\
& + 2\frac{d\phi^3}{dn}\left(\frac{dU}{dm}\Omega^\alpha_\gamma m^\gamma - \frac{D\Omega^\alpha_\gamma}{dz}n^\gamma + L_{nm}\Omega^\alpha_\gamma m^\gamma\right)\Bigg\}\vec{h}_\alpha \\
+ \Bigg\{ & \frac{d^3\phi^3}{dz\,dn^2} - \frac{d\phi^3}{dz}\Omega_{\alpha\theta}\Omega^\theta_\beta n^\alpha n^\beta - 2\frac{d\phi^3}{dn}\Omega_{\alpha\gamma}n^\gamma(L^{3\alpha} + U\Omega^\alpha_\beta n^\beta) \\
& + 2\frac{D^2\phi^\alpha}{dz\,Dn}\Omega_{\alpha\gamma}n^\gamma + \frac{D^2\phi^\alpha}{Dn^2}(L^3{}_\alpha + U\Omega_{\alpha\gamma}n^\gamma) + \frac{D\phi^\alpha}{dz}\left(\frac{D\Omega_{\alpha\gamma}}{Dn}n^\gamma + \mu\Omega_{\alpha\gamma}m^\gamma\right) \\
& - 2\frac{D\phi^\alpha}{Dn}\left(\frac{dU}{dm}\Omega_{\alpha\gamma}m^\gamma - \frac{D\Omega_{\alpha\gamma}}{dz}n^\gamma + L_{nm}\Omega_{\alpha\gamma}m^\gamma\right)\Bigg\}\vec{h}_3.
\end{aligned}
\tag{4.27}
$$

From (4.22) differentiated again and with the usual simplifications,

$$
\begin{aligned}
\frac{d^3\vec{\phi}}{dz^2\,dn} \doteq & \left(\frac{D^3\phi^\alpha}{dz^2\,Dn} - \frac{d^2\phi^3}{dz\,dn}L^{3\alpha} - \frac{d\phi^3}{dn}\frac{DL^{3\alpha}}{dz} - \frac{dU}{dz}\frac{d\phi^3}{dn}\Omega^\alpha_\gamma n^\gamma\right. \\
& \left. - U\frac{d^2\phi^3}{dz\,dn}\Omega^\alpha_\gamma n^\gamma - U\frac{d\phi^3}{dn}\frac{D\Omega^\alpha_\gamma}{dz}n^\gamma + U\frac{d\phi^3}{dn}\Omega^\alpha_\gamma\left(\frac{dU}{dm} + L_{nm}\right)m^\gamma\right)\vec{h}_\alpha \\
& - \left(\frac{d^2\phi^3}{dz\,dn} + \frac{D\phi^\delta}{Dn}L^3{}_\delta + U\frac{D\phi^\delta}{Dn}\Omega_{\delta\gamma}n^\gamma\right)(L^{3\alpha} + Un^\lambda\Omega^\alpha_\lambda)\vec{h}_\alpha \\
& + \left(\frac{d^3\phi^3}{dz^2\,dn} + \frac{D^2\phi^\delta}{dz\,Dn}L^3{}_\delta + \frac{D\phi^\delta}{Dn}\frac{DL^3{}_\delta}{dz} + \frac{dU}{dz}\frac{D\phi^\delta}{Dn}\Omega_{\delta\gamma}n^\gamma\right. \\
& \left. + U\frac{D^2\phi^\delta}{dz\,Dn}\Omega_{\delta\gamma}n^\gamma + U\frac{D\phi^\delta}{Dn}\frac{D\Omega_{\delta\gamma}}{dz}n^\gamma + U\frac{D\phi^\delta}{Dn}\Omega_{\delta\gamma}\left(\frac{dU}{dm} + L_{nm}\right)m^\gamma\right)\vec{h}_3 \\
& + \left(\frac{D^2\phi^\alpha}{dz\,Dn} - \frac{d\phi^3}{dn}L^{3\alpha} - U\frac{d\phi^3}{dn}\Omega^\alpha_\gamma n^\gamma\right)(L^3{}_\alpha + Un^\lambda\Omega_{\alpha\lambda})\vec{h}_3.
\end{aligned}
\tag{4.28}
$$

5 Partial Derivatives

We take the second covariant derivative (3.1) of the vector field from [8],

$$
\begin{aligned}
\vec{\phi}_{|\alpha\beta} &= \vec{\phi}_{,\alpha\beta} - \vec{\phi}_{,\delta}\left\{\begin{matrix}\delta\\ \alpha\beta\end{matrix}\right\} \\
&= (\phi^\lambda_{|\alpha\beta} - \phi^\sigma\Omega_{\sigma\alpha}\Omega^\lambda_\beta - \phi^3_{,\alpha}\Omega^\lambda_\beta - \phi^3_{,\beta}\Omega^\lambda_\alpha - \phi^3\Omega^\lambda_{\alpha|\beta})\vec{h}_\lambda \\
&\quad + (\phi^\lambda_{|\alpha}\Omega_{\lambda\beta} + (\phi^\lambda\Omega_{\lambda\alpha})_{|\beta} + \phi^3_{|\alpha\beta} - \phi^3\Omega^\lambda_\alpha\Omega_{\lambda\beta})\vec{h}_3.
\end{aligned}
\tag{5.1}
$$

From this we obtain directly, using (2.7) and (2.8) above,

$$\begin{aligned}\dot{\overrightarrow{\phi}}_{|\beta\gamma} =&(\dot{\overline{\phi^{\alpha}_{|\beta\gamma}}} - \dot\phi^\sigma\Omega_{\sigma\beta}\Omega^\alpha_\gamma - \phi^\sigma\dot\Omega_{\sigma\beta}\Omega^\alpha_\gamma - \phi^\sigma\Omega_{\sigma\beta}\dot\Omega^\alpha_\gamma\\
&- \phi^\delta_{|\beta}L^{3\alpha}\Omega_{\delta\gamma} - (\phi^\delta\Omega_{\delta\beta})_{|\gamma}L^{3\alpha} + \phi^\delta_{|\beta\gamma}L^\alpha{}_\delta - \phi^\sigma L^\alpha{}_\delta\Omega_{\sigma\beta}\Omega^\delta_\gamma\\
&- \dot\phi^3_{,\beta}\Omega^\alpha_\gamma - \phi^3_{,\beta}\dot\Omega^\alpha_\gamma - \dot\phi^3_{,\gamma}\Omega^\alpha_\beta - \phi^3_{,\gamma}\dot\Omega^\alpha_\beta - \dot\phi^3\Omega^\alpha_{\beta|\gamma}\\
&- \phi^3\dot{\overline{\Omega^\alpha_{\beta|\gamma}}} - \phi^3_{|\beta\gamma}L^{3\alpha} + \phi^3L^{3\alpha}\Omega^\delta_\beta\Omega_{\delta\gamma} - \phi^3_{,\beta}L^\alpha{}_\delta\Omega^\delta_\gamma - (\phi^3\Omega^\delta_\beta)_{|\gamma}L^\alpha{}_\delta)\vec h_\alpha\\
&+ (\phi^\delta_{|\beta\gamma}L^3{}_\delta - \phi^\sigma L^3{}_\delta\Omega_{\sigma\beta}\Omega^\delta_\gamma + \dot{\overline{\phi^\alpha_{|\beta}}}\Omega_{\alpha\gamma} + \phi^\alpha_{|\beta}\dot\Omega_{\alpha\gamma}\\
&+ \dot{\overline{\phi^\alpha_{|\gamma}}}\Omega_{\alpha\beta} + \phi^\alpha_{|\gamma}\dot\Omega_{\alpha\beta} + \dot\phi^\alpha\Omega_{\alpha\beta|\gamma} + \phi^\alpha\dot{\overline{\Omega_{\alpha\beta|\gamma}}}\\
&- \phi^3_{,\beta}L^3{}_\delta\Omega^\delta_\gamma - (\phi^3\Omega^\delta_\beta)_{|\gamma}L^3{}_\delta + \dot{\overline{\phi^3_{|\beta\gamma}}} - \dot\phi^3\Omega^\alpha_\beta\Omega_{\alpha\gamma} - \phi^3\dot\Omega^\alpha_\beta\Omega_{\alpha\gamma} - \phi^3\Omega^\alpha_\beta\dot\Omega_{\alpha\gamma})\vec h_3.\end{aligned} \tag{5.2}$$

In order to be able to compare this expression with one based on directional derivatives, three transformations need to be applied to it. All of the dotted ϕ terms except the two that formulas are sought for, $\dot{\overline{\phi^\alpha_{|\beta\gamma}}}$ and $\dot{\overline{\phi^3_{|\beta\gamma}}}$ must be replaced. This replacement is done for first derivatives by using the expressions (4.12) and, where an expression is a second derivative, using the expressions from which the second-order kinematic compatibility conditions come when jumps are taken, (4.11). The factor $\dot{\overline{\phi^\alpha_{|\beta}}}$ can be dealt with exactly like $\dot\phi^\alpha_{|\beta}$ because the difference between the two expressions, according to equation (8.30) of [6], is ϕ^ϵ times a geometrical coefficient and so can be ignored here since its jump will vanish. The second transformation is to expand covariant derivatives in terms of directional derivatives using formulas

$$\phi^\alpha_{|\beta} = \frac{D\phi^\alpha}{Dm}m_\beta + \frac{D\phi^\alpha}{Dn}n_\beta, \quad \phi^3_{,\beta} = \frac{d\phi^3}{dm}m_\beta + \frac{d\phi^3}{dn}n_\beta, \tag{5.3}$$

and the same iterated and similar formulas applied to Ω^α_γ, $L^{3\alpha}$, and $L^\alpha{}_\gamma$. The third transformation is to insert components of the tangent and normal vectors $\vec m$ and $\vec n$ in each term except $\dot{\overline{\phi^\alpha_{|\beta\gamma}}}$ and $\dot{\overline{\phi^3_{|\beta\gamma}}}$. This can be done by replacing each subscript β (or γ respectively) by a dummy suffix, say ϵ, and multiplying the term by δ^ϵ_β in the form $m^\epsilon m_\beta + n^\epsilon n_\beta$ (equation (6.25) of [6]). The result of these expansions is a substantial expression, which is not worth looking at and which was formed by the *Mathematica* package 'Ricci' of Jack Lee running on a NeXT computer. (Without this combination of facilities, these calculations would not have been completed.)

We take the second partial derivative of $\vec\phi$ with respect to time from [7],

$$\begin{aligned}\ddot{\vec\phi} = &(\ddot\phi^\alpha + 2\dot\phi^\gamma L^\alpha{}_\gamma + \phi^\gamma\dot L^\alpha{}_\gamma + \phi^\lambda L^\gamma{}_\lambda L^\alpha{}_\gamma - \phi^\gamma L^3{}_\gamma L^{3\alpha}\\
&- 2\dot\phi^3L^{3\alpha} - \phi^3\dot L^{3\alpha} - \phi^3L^{3\gamma}L^\alpha{}_\gamma)\vec h_\alpha\\
&+ (\ddot\phi^3 + 2\dot\phi^\gamma L^3{}_\gamma + \phi^\gamma\dot L^3{}_\gamma - \phi^3L^{3\gamma}L^3{}_\gamma + \phi^\lambda L^\gamma{}_\lambda L^3{}_\gamma)\vec h_3,\end{aligned} \tag{5.4}$$

and differentiate it partially with respect to θ^λ to obtain

$$\begin{aligned}\ddot{\vec\phi}_{,\lambda} = &(\ddot\phi^\alpha_{|\lambda} + 2\dot\phi^\gamma_{|\lambda}L^\alpha{}_\gamma + 2\dot\phi^\gamma L^\alpha{}_{\gamma|\lambda} + \phi^\gamma_{|\lambda}\dot L^\alpha{}_\gamma + \phi^\gamma\dot L^\alpha{}_{\gamma|\lambda} + \phi^\delta_{|\lambda}L^\gamma{}_\delta L^\alpha{}_\gamma\\
&+ \phi^\delta L^\gamma{}_{\delta|\lambda}L^\alpha{}_\gamma + \phi^\delta L^\gamma{}_\delta L^\alpha{}_{\gamma|\lambda} - \phi^\gamma_{|\lambda}L^3{}_\gamma L^{3\alpha} - \phi^\gamma L^3{}_{\gamma|\lambda}L^{3\alpha} - \phi^\gamma L^3{}_\gamma L^{3\alpha}{}_{|\lambda}\\
&- 2\dot\phi^3_{,\lambda}L^{3\alpha} - 2\dot\phi^3L^{3\alpha}_{|\lambda} - \phi^3_{,\lambda}\dot L^{3\alpha} - \phi^3\dot L^{3\alpha}{}_{|\lambda} - \phi^3_{,\lambda}L^{3\gamma}L^\alpha{}_\gamma - \phi^3L^{3\gamma}{}_{|\lambda}L^\alpha{}_\gamma - \phi^3L^{3\gamma}L^\alpha{}_{\gamma|\lambda})\vec h_\alpha\\
&- (2\dot\phi^\gamma L^3{}_\gamma + \phi^\gamma\dot L^3{}_\gamma + \phi^\delta L^\gamma{}_\delta L^3{}_\gamma + \ddot\phi^3 - \phi^3L^{3\gamma}L^3{}_\gamma)\Omega^\alpha_\lambda\vec h_\alpha\\
&+ (\ddot\phi^\alpha + 2\dot\phi^\gamma L^\alpha{}_\gamma + \phi^\gamma\dot L^\alpha{}_\gamma + \phi^\delta L^\gamma{}_\delta L^\alpha{}_\gamma - \phi^\gamma L^3{}_\gamma L^{3\alpha} - 2\dot\phi^3L^{3\alpha} - \phi^3\dot L^{3\alpha} - \phi^3L^{3\gamma}L^\alpha{}_\gamma)\Omega_{\alpha\lambda}\vec h_3\\
&+ (2\dot\phi^\gamma_{|\lambda}L^3{}_\gamma + 2\dot\phi^\gamma L^3{}_{\gamma|\lambda} + \phi^\gamma_{|\lambda}\dot L^3{}_\gamma + \phi^\gamma\dot L^3{}_{\gamma|\lambda} + \phi^\delta_{|\lambda}L^\gamma{}_\delta L^3{}_\gamma + \phi^\delta L^\gamma{}_{\delta|\lambda}L^3{}_\gamma + \phi^\delta L^\gamma{}_\delta L^3{}_{\gamma|\lambda}\\
&+ \ddot\phi^3_{,\lambda} - \phi^3_{,\lambda}L^{3\gamma}L^3{}_\gamma - \phi^3L^{3\gamma}_{|\lambda}L^3{}_\gamma - \phi^3L^{3\gamma}L^3{}_{\gamma|\lambda})\vec h_3.\end{aligned} \tag{5.5}$$

Since this expression will not again be differentiated, it is permissible to simplify it by the elimination of terms containing ϕ^α and ϕ^3, of which there are a goodly number. For comparison with the corresponding expression based on directional derivatives, as above the dotted ϕ terms except those sought, $\ddot{\phi}^\alpha_{|\lambda}$ and $\ddot{\phi}^3_{|\lambda}$ need to be replaced and the covariant derivatives expanded in directional derivatives. The result of this expansion is again a substantial expression, which exists only in the format of 'Ricci'.

The third of the partial derivatives needed is obtained by differentiating the expression (5.4) with respect to time to obtain,

$$\begin{aligned}
\dddot{\vec{\phi}} &= (\dddot{\phi}^\alpha + 2\ddot{\phi}^\gamma L^\alpha{}_\gamma + 2\dot{\phi}^\gamma \dot{L}^\alpha{}_\gamma + \dot{\phi}^\gamma \dot{L}^\alpha{}_\gamma + \phi^\gamma \ddot{L}^\alpha{}_\gamma + \dot{\phi}^\lambda L^\gamma{}_\lambda L^\alpha{}_\gamma \\
&\quad + \phi^\lambda \dot{L}^\gamma{}_\lambda L^\alpha{}_\gamma + \phi^\lambda L^\gamma{}_\lambda \dot{L}^\alpha{}_\gamma - \dot{\phi}^\gamma L^3{}_\gamma L^{3\alpha} - \phi^\gamma \dot{L}^3{}_\gamma L^{3\alpha} - \phi^\gamma L^3{}_\gamma \dot{L}^{3\alpha} - 2\ddot{\phi}^3 L^{3\alpha} \\
&\quad - 2\dot{\phi}^3 \dot{L}^{3\alpha} - \dot{\phi}^3 \dot{L}^{3\alpha} - \phi^3 \ddot{L}^{3\alpha} - \dot{\phi}^3 L^{3\gamma} L^\alpha{}_\gamma - \phi^3 \dot{L}^{3\gamma} L^\alpha{}_\gamma - \phi^3 L^{3\gamma} \dot{L}^\alpha{}_\gamma)\vec{h}_\alpha \\
&+ (\ddot{\phi}^3 + 2\dot{\phi}^\gamma L^3{}_\gamma + \phi^\gamma \dot{L}^3{}_\gamma - \phi^3 L^{3\gamma} L^3{}_\gamma + \phi^\lambda L^\gamma{}_\lambda L^3{}_\gamma)(-L^{3\alpha}\vec{h}_\alpha) \\
&+ (\ddot{\phi}^\delta + 2\dot{\phi}^\gamma L^\delta{}_\gamma + \phi^\gamma \dot{L}^\delta{}_\gamma + \phi^\lambda L^\gamma{}_\lambda L^\delta{}_\gamma - \phi^\gamma L^3{}_\gamma L^{3\delta} - 2\dot{\phi}^3 L^{3\delta} - \phi^3 \dot{L}^{3\delta} - \phi^3 L^{3\gamma} L^\delta{}_\gamma) L^\alpha{}_\delta \vec{h}_\alpha \\
&+ (\ddot{\phi}^\delta + 2\dot{\phi}^\gamma L^\delta{}_\gamma + \phi^\gamma \dot{L}^\delta{}_\gamma + \phi^\lambda L^\gamma{}_\lambda L^\delta{}_\gamma - \phi^\gamma L^3{}_\gamma L^{3\delta} - 2\dot{\phi}^3 L^{3\delta} - \phi^3 \dot{L}^{3\delta} - \phi^3 L^{3\gamma} L^\delta{}_\gamma) L^3{}_\delta \vec{h}_3 \\
&+ (\dddot{\phi}^3 - \dot{\phi}^3 L^{3\gamma} L^3{}_\gamma - \phi^3 \dot{L}^{3\gamma} L^3{}_\gamma - \phi^3 L^{3\gamma} \dot{L}^3{}_\gamma + 2\ddot{\phi}^\gamma L^3{}_\gamma + 3\dot{\phi}^\gamma \dot{L}^3{}_\gamma \\
&+ \phi^\gamma \ddot{L}^3{}_\gamma + \dot{\phi}^\lambda L^\gamma{}_\lambda L^3{}_\gamma + \phi^\lambda \dot{L}^\gamma{}_\lambda L^3{}_\gamma + \phi^\lambda L^\gamma{}_\lambda \dot{L}^3{}_\gamma)\vec{h}_3.
\end{aligned} \tag{5.6}$$

Again terms containing ϕ^α and ϕ^3 can legitimately be removed to give the simplified form

$$\begin{aligned}
\dddot{\vec{\phi}} &\doteq (\dddot{\phi}^\alpha + 3\ddot{\phi}^\gamma L^\alpha{}_\gamma + 3\dot{\phi}^\gamma \dot{L}^\alpha{}_\gamma + 3\dot{\phi}^\lambda L^\gamma{}_\lambda L^\alpha{}_\gamma - 3\dot{\phi}^\gamma L^3{}_\gamma L^{3\alpha} \\
&\quad - 3\ddot{\phi}^3 L^{3\alpha} - 3\dot{\phi}^3 \dot{L}^{3\alpha} - 3\dot{\phi}^3 L^{3\gamma} L^\alpha{}_\gamma)\vec{h}_\alpha \\
&\quad + (\dddot{\phi}^3 - 3\dot{\phi}^3 L^{3\gamma} L^3{}_\gamma + 3\ddot{\phi}^\gamma L^3{}_\gamma + 3\dot{\phi}^\gamma \dot{L}^3{}_\gamma + 3\dot{\phi}^\lambda L^\gamma{}_\lambda L^3{}_\gamma)\vec{h}_3.
\end{aligned} \tag{5.7}$$

This expression then requires the same expansion as the previous partial derivative to produce an expression for comparison with the directional-derivative version of the same quantity. This has been done in 'Ricci'.

6 Compatibility Conditions

The expression (3.18) for $\dot{\overline{\vec{\phi}_{|\alpha\beta}}}$ in terms of directional derivatives, with each vectorial expression expanded into co-ordinates by the formulas stated in section two or derived in section four and the now usual simplification, can be compared piece by piece with the same quantity computed in the fifth section to give expressions for $\dot{\overline{\phi^\alpha_{|\beta\gamma}}}$ and $\dot{\overline{\phi^3_{|\beta\gamma}}}$, which appear in that form in the section-five formulas. The resulting expressions, true only when jumps are taken, are:

$$
\begin{aligned}
\dot{\overline{\phi^\alpha_{|\beta\gamma}}} \doteq \Big\{ & \eta U \frac{D^2\phi^\alpha}{Dn^2} - U\frac{D^3\phi^\alpha}{Dm^2\,Dn} - 2\frac{dU}{dm}\frac{D^2\phi^\alpha}{Dm\,Dn} - \eta\frac{D^2\phi^\alpha}{dz\,Dn} \\
& + \left(KU + \eta^2 U - \frac{d\eta}{dz} - 2\eta L_{mm}\right)\frac{D\phi^\alpha}{Dn} + \eta L^{3\alpha}\frac{d\phi^3}{dn}\Big\} m_\beta m_\gamma \\
& + \Big\{ \frac{D^3\phi^\alpha}{dz\,Dm\,Dn} - U\frac{D^3\phi^\alpha}{Dm\,Dn^2} - \frac{dU}{dm}\frac{D^2\phi^\alpha}{Dn^2} - L^\alpha{}_\delta\frac{D^2\phi^\delta}{Dm\,Dn} \\
& + (L_{mm} + L_{nn} - 2\eta U)\frac{D^2\phi^\alpha}{Dm\,Dn} - \eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D\phi^\alpha}{Dn} \\
& + \{2U m^\delta n^\epsilon(\Omega_{\lambda\epsilon}\Omega^\alpha_\delta - \Omega_{\lambda\delta}\Omega^\alpha_\epsilon) + \mu L^\alpha{}_\lambda\}\frac{D\phi^\lambda}{Dn} + (M^\alpha - \mu L^{3\alpha})\frac{d\phi^3}{dn}\Big\} m_\beta n_\gamma \\
& + \Big\{ \frac{D^3\phi^\alpha}{dz\,Dm\,Dn} - U\frac{D^3\phi^\alpha}{Dm\,Dn^2} - \frac{dU}{dm}\frac{D^2\phi^\alpha}{Dn^2} - L^\alpha{}_\delta\frac{D^2\phi^\delta}{Dm\,Dn} \\
& + (L_{mm} + L_{nn} - 2\eta U)\frac{D^2\phi^\alpha}{Dm\,Dn} - \eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D\phi^\alpha}{Dn} \\
& + \{U m^\delta n^\epsilon(\Omega_{\lambda\epsilon}\Omega^\alpha_\delta - \Omega_{\lambda\delta}\Omega^\alpha_\epsilon) + \mu L^\alpha{}_\lambda\}\frac{D\phi^\lambda}{Dn} + (M^\alpha - \eta L^{3\alpha})\frac{d\phi^3}{dn}\Big\} n_\beta m_\gamma \\
& + \Big\{ \frac{D^3\phi^\alpha}{dz\,Dn^2} - U\frac{D^3\phi^\alpha}{Dn^3} + \left(2\frac{dU}{dm} + 3\mu U + 2L_{mn} + 2L_{nm}\right)\frac{D^2\phi^\alpha}{Dm\,Dn} \\
& + 2L_{nn}\frac{D^2\phi^\alpha}{Dn^2} - L^\alpha{}_\delta\frac{D^2\phi^\delta}{Dn^2} - \mu^2 U\frac{D\phi^\alpha}{Dn} - \mu L^\alpha{}_\delta\frac{D\phi^\delta}{Dn} + (2N^\alpha + \mu L^{3\alpha})\frac{d\phi^3}{dn}\Big\} n_\beta n_\gamma,
\end{aligned}
\tag{6.1}
$$

where

$$
\begin{aligned}
M^\alpha &= \frac{DL^{3\alpha}}{Dm} - L_{mm}\Omega^\alpha_\beta m^\beta - L_{nm}\Omega^\alpha_\beta n^\beta - \frac{D\Omega^\alpha_\beta}{dz}m^\beta + U\frac{D\Omega^\alpha_\beta}{Dm}n^\beta, \\
N^\alpha &= \frac{DL^{3\alpha}}{Dn} - L_{mn}\Omega^\alpha_\beta m^\beta - L_{nn}\Omega^\alpha_\beta n^\beta - \frac{D\Omega^\alpha_\beta}{dz}n^\beta + U\frac{D\Omega^\alpha_\gamma}{Dn}n^\gamma,
\end{aligned}
\tag{6.2}
$$

both vanish, as was shown in [7], and

$$
\begin{aligned}
\dot{\overline{\phi^3_{|\beta\gamma}}} \doteq \Big\{ & \eta U \frac{d^2\phi^3}{dn^2} - U\frac{d^3\phi^3}{dm^2\,dn} - 2\frac{dU}{dm}\frac{d^2\phi^3}{dm\,dn} - \eta\frac{d^2\phi^3}{dz\,dn} \\
& + \left(KU + \eta^2 U - \frac{d\eta}{dz} - 2\eta L_{mm} \right) \frac{d\phi^3}{dn} - \eta L^3{}_\alpha \frac{D\phi^\alpha}{Dn} \Big\} m_\beta m_\gamma \\
& + \Big\{ \frac{d^3\phi^3}{dz\,dm\,dn} - U\frac{d^3\phi^3}{dm\,dn^2} - \frac{dU}{dm}\frac{d^2\phi^3}{dn^2} \\
& + (L_{mm} + L_{nn} - 2\eta U)\frac{d^2\phi^3}{dm\,dn} - \eta\left(\frac{dU}{dm} + L_{mn} + L_{nm} \right)\frac{d\phi^3}{dn} \\
& + (m^\delta P_{\alpha\delta} - M_\alpha + \mu L^3{}_\alpha)\frac{D\phi^\alpha}{Dn} \Big\} m_\beta n_\gamma \\
& + \Big\{ \frac{d^3\phi^3}{dz\,dm\,dn} - U\frac{d^3\phi^3}{dm\,dn^2} - \frac{dU}{dm}\frac{d^2\phi^3}{dn^2} \\
& + (L_{mm} + L_{nn} - 2\eta U)\frac{d^2\phi^3}{dm\,dn} - \eta\left(\frac{dU}{dm} + L_{mn} + L_{nm} \right)\frac{d\phi^3}{dn} \\
& + (m^\delta P_{\alpha\delta} - M_\alpha + \eta L^3{}_\alpha)\frac{D\phi^\alpha}{Dn} \Big\} n_\beta m_\gamma \\
& + \Big\{ \frac{d^3\phi^3}{dz\,dn^2} - U\frac{d^3\phi^3}{dn^3} + \left(2\frac{dU}{dm} + 3\mu U + 2L_{mn} + 2L_{nm} \right)\frac{d^2\phi^3}{dm\,dn} \\
& + 2L_{nn}\frac{d^2\phi^3}{dn^2} - \mu^2 U\frac{d\phi^3}{dn} \\
& + \left(2(n^\delta P_{\alpha\delta} - N_\alpha) - \mu L^3{}_\alpha \right)\frac{D\phi^\alpha}{Dn} \Big\} n_\beta n_\gamma,
\end{aligned}
\tag{6.3}
$$

where

$$
P_{\alpha\delta} = \Omega_{\delta\epsilon} L^\epsilon{}_\alpha - \Omega_{\alpha\epsilon} L^\epsilon{}_\delta - m_\alpha \frac{DL^3{}_\delta}{Dm} - n_\alpha \frac{DL^3{}_\delta}{Dn} + L^3{}_{\alpha|\delta}. \tag{6.4}
$$

To see that $P_{\alpha\delta} = 0$ one combines its directional derivatives of $L^3{}_\delta$ as $L^3{}_{\delta|\alpha}$ and then expands the covariant derivatives in terms of the definition of $L^3{}_\alpha$ (2.3). The whole expression then vanishes because v^3 is a scalar and $\Omega_{\alpha\beta}$, $\Omega_{\gamma\alpha|\beta}$, and $\Omega_{\alpha\gamma}\Omega^\gamma_\beta$ are symmetric in α and β.

When jumps are taken, a further simplification comes from using (4.7_2) and (4.18);

$$
\begin{aligned}
\left[\dot{\overline{\phi^3_{|\beta\gamma}}}\right] \doteq \Big\{ & \eta U\tilde{\tilde{\psi}}^3 - U\frac{d^2\tilde{\psi}^3}{dm^2} - 2\frac{dU}{dm}\frac{d\tilde{\psi}^3}{dm} - \eta\frac{d\tilde{\psi}^3}{dz} \\
& + \left[\left(KU + \eta^2 U - \frac{d\eta}{dz} - 2\eta L_{mm} \right)\frac{d\phi^3}{dn} \right] \Big\} m_\beta m_\gamma \\
& + \Big\{ \frac{d^2\tilde{\psi}^3}{dz\,dm} - U\frac{d\tilde{\tilde{\psi}}^3}{dm} - \frac{dU}{dm}\tilde{\tilde{\psi}}^3 + \left[(L_{mm} + L_{nn} - 2\eta U)\frac{d^2\phi^3}{dm\,dn} \right] \\
& - \eta\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm} \right)\frac{d\phi^3}{dn} \right] \Big\} (m_\beta n_\gamma + n_\beta m_\gamma) \\
& + \Big\{ \frac{d\tilde{\tilde{\psi}}^3}{dz} - U\tilde{\tilde{\tilde{\psi}}}^3 + 2\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm} \right)\frac{d^2\phi^3}{dm\,dn} \right] \\
& + 2\left[L_{nn}\frac{d^2\phi^3}{dn^2} \right] - \mu\left[L^3{}_\alpha \frac{D\phi^\alpha}{Dn} \right] \Big\} n_\beta n_\gamma \\
& + \left[L^3{}_\alpha \frac{D\phi^\alpha}{Dn} \right] (-\eta m_\beta m_\gamma + \mu m_\beta n_\gamma + \eta n_\beta m_\gamma - \mu n_\beta n_\gamma).
\end{aligned}
\tag{6.5}
$$

The tangential formula is not subject to the $P_{\alpha\beta}$ simplification, but only (4.7_1) and (4.17). So

$$\begin{aligned}
\left[\dot{\overline{\phi^{\alpha}_{|\beta\gamma}}}\right] \doteq & \left\{\eta U\tilde{\tilde{\psi}}^{\alpha} - U\frac{D^2\tilde{\psi}^{\alpha}}{Dm^2} - 2\frac{dU}{dm}\frac{D\tilde{\psi}^{\alpha}}{Dm} - \eta\frac{D\tilde{\psi}^{\alpha}}{dz}\right. \\
& \left. + \left[\left(KU + \eta^2 U - \frac{d\eta}{dz} - 2\eta L_{mm}\right)\frac{D\phi^{\alpha}}{Dn}\right]\right\} m_{\beta}m_{\gamma} \\
& + \left\{\frac{D^2\tilde{\psi}^{\alpha}}{dz\,Dm} - U\frac{D\tilde{\tilde{\psi}}^{\alpha}}{Dm} - \frac{dU}{dm}\tilde{\tilde{\psi}}^{\alpha} - \left[L^{\alpha}{}_{\delta}\frac{D^2\phi^{\delta}}{Dm\,Dn}\right]\right. \\
& \quad + \left[(L_{mm} + L_{nn} - 2\eta U)\frac{D^2\phi^{\alpha}}{Dm\,Dn}\right] - \eta\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D\phi^{\alpha}}{Dn}\right] \\
& \quad \left. + \left[(Um^{\delta}n^{\epsilon}R^{\alpha}{}_{\lambda\delta\epsilon} + \mu L^{\alpha}{}_{\lambda})\frac{D\phi^{\lambda}}{Dn}\right]\right\}(m_{\beta}n_{\gamma} + n_{\beta}m_{\gamma}) \qquad (6.6) \\
& + Um^{\delta}n^{\epsilon}\left[R^{\alpha}{}_{\lambda\delta\epsilon}\frac{D\phi^{\lambda}}{Dn}\right] m_{\beta}n_{\gamma} \\
& + \left\{\frac{D\tilde{\tilde{\psi}}^{\alpha}}{dz} - U\tilde{\tilde{\psi}}^{\alpha} + 2\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D^2\phi^{\alpha}}{Dm\,Dn}\right]\right. \\
& \quad \left. + 2\left[L_{nn}\frac{D^2\phi^{\alpha}}{Dn^2}\right] - \left[L^{\alpha}{}_{\delta}\frac{D^2\phi^{\delta}}{Dn^2}\right] - \mu\left[L^{\alpha}{}_{\delta}\frac{D\phi^{\delta}}{Dn}\right]\right\} n_{\beta}n_{\gamma} \\
& + \left[L^{3\alpha}\frac{d\phi^3}{dn}\right](\eta m_{\beta}m_{\gamma} - \mu m_{\beta}n_{\gamma} - \eta n_{\beta}m_{\gamma} + \mu n_{\beta}n_{\gamma}).
\end{aligned}$$

For the remaining formulas, it is necessary to have $\ddot{\vec{\phi}}_{,\lambda}$ in terms of directional derivatives. This is obtained by partial differentiation with respect to t of $\dot{\vec{\phi}}_{,\lambda}$, which has already been used above. From (3.19), performing the partial differentiation using z and n, the result is

$$
\begin{aligned}
\ddot{\vec{\phi}}_{,\lambda} &= \left(\frac{d}{dz} - U\frac{d}{dn}\right)\left\{\frac{d^2\vec{\phi}}{dzdm} - U\frac{d^2\vec{\phi}}{dmdn} + (L_{mm} - U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\}m_\lambda \\
&\quad + \left(\frac{d}{dz} - U\frac{d}{dn}\right)\left\{\frac{d^2\vec{\phi}}{dzdn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\}n_\lambda \\
&\quad + \left\{\frac{d^2\vec{\phi}}{dzdm} - U\frac{d^2\vec{\phi}}{dmdn} + (L_{mm} - U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\}\left(\frac{dm_\lambda}{dz} - U\frac{dm_\lambda}{dn}\right) \\
&\quad + \left\{\frac{d^2\vec{\phi}}{dzdn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\}\left(\frac{dn_\lambda}{dz} - U\frac{dn_\lambda}{dn}\right) \\
&= \left\{\frac{d^3\vec{\phi}}{dz^2dm} - \frac{dU}{dz}\frac{d^2\vec{\phi}}{dmdn} - U\frac{d^3\vec{\phi}}{dzdmdn} + \frac{d}{dz}(L_{mm} - U\eta)\frac{d\vec{\phi}}{dm}\right. \\
&\quad \left. + (L_{mm} - U\eta)\frac{d^2\vec{\phi}}{dzdm} - \frac{d^2U}{dzdm}\frac{d\vec{\phi}}{dn} - \frac{dU}{dm}\frac{d^2\vec{\phi}}{dzdn}\right\}m_\lambda \\
&\quad - U\frac{d}{dn}\left\{\frac{d^2\vec{\phi}}{dzdm} - U\frac{d^2\vec{\phi}}{dmdn} + (L_{mm} - U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\}m_\lambda \\
&\quad + \left\{\frac{d^3\vec{\phi}}{dz^2dn} - \frac{dU}{dz}\frac{d^2\vec{\phi}}{dn^2} - U\frac{d^3\vec{\phi}}{dzdn^2} + \frac{d}{dz}\left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm}\right. \\
&\quad \left. + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d^2\vec{\phi}}{dzdm} + \frac{dL_{nn}}{dz}\frac{d\vec{\phi}}{dn} + L_{nn}\frac{d^2\vec{\phi}}{dzdn}\right\}n_\lambda \\
&\quad - U\frac{d}{dn}\left\{\frac{d^2\vec{\phi}}{dzdn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\}n_\lambda \\
&\quad + \left\{\frac{d^2\vec{\phi}}{dzdm} - U\frac{d^2\vec{\phi}}{dmdn} + (L_{mm} - U\eta)\frac{d\vec{\phi}}{dm} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right\}\left(L_{mm}m_\lambda + \big(L_{mn} + L_{nm} + \frac{dU}{dm} + U\mu\big)n_\lambda\right) \\
&\quad + \left\{\frac{d^2\vec{\phi}}{dzdn} - U\frac{d^2\vec{\phi}}{dn^2} + \left(\frac{dU}{dm} + L_{nm} + L_{mn} + U\mu\right)\frac{d\vec{\phi}}{dm} + L_{nn}\frac{d\vec{\phi}}{dn}\right\}\left(-\big(U\mu + \frac{dU}{dm}\big)m_\lambda + L_{nn}n_\lambda\right).
\end{aligned}
\tag{6.7}
$$

In the above expression, lines three and six can be replaced at a stroke using the pair of commutation relations (3.23). The expression can be simplified by the removal of terms involving differentiation of $\vec{\phi}$ only in the directions $\vec{m}$, $\vec{z}$, or a combination of them;

$$
\begin{aligned}
\ddot{\vec{\phi}}_{,\lambda} \doteq & \left(-\frac{dU}{dz}\frac{d^2\vec{\phi}}{dmdn} - U\frac{d^3\vec{\phi}}{dzdmdn} - \frac{d^2U}{dzdm}\frac{d\vec{\phi}}{dn} - \frac{dU}{dm}\frac{d^2\vec{\phi}}{dzdn}\right)m_\lambda \\
& + \left(\frac{d^3\vec{\phi}}{dz^2dn} - \frac{dU}{dz}\frac{d^2\vec{\phi}}{dn^2} - U\frac{d^3\vec{\phi}}{dzdn^2} + \frac{dL_{nn}}{dz}\frac{d\vec{\phi}}{dn} + L_{nn}\frac{d^2\vec{\phi}}{dzdn}\right)n_\lambda \\
& + \left(-U\frac{d^2\vec{\phi}}{dmdn} - \frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right)\left(L_{mm}m_\lambda + \left(L_{mn} + L_{nm} + \frac{dU}{dm} + U\mu\right)n_\lambda\right) \\
& + \left(\frac{d^2\vec{\phi}}{dzdn} - U\frac{d^2\vec{\phi}}{dn^2} + L_{nn}\frac{d\vec{\phi}}{dn}\right)\left\{-\left(U\mu + \frac{dU}{dm}\right)m_\lambda + L_{nn}n_\lambda\right\} \\
& - U\left\{\left(\frac{d^3\vec{\phi}}{dzdmdn} - U\frac{d^3\vec{\phi}}{dmdn^2} - 2U\eta\frac{d^2\vec{\phi}}{dmdn} + L_{mm}\frac{d^2\vec{\phi}}{dmdn} + L_{nn}\frac{d^2\vec{\phi}}{dmdn}\right.\right. \\
& \quad \left. - \frac{dU}{dm}\frac{d^2\vec{\phi}}{dn^2} - L_{mn}\eta\frac{d\vec{\phi}}{dn} - L_{nm}\eta\frac{d\vec{\phi}}{dn} - \frac{dU}{dm}\eta\frac{d\vec{\phi}}{dn}\right)m_\lambda \\
& \quad - \mu\left(\frac{d^2\vec{\phi}}{dzdn} - U\frac{d^2\vec{\phi}}{dn^2} + L_{nn}\frac{d\vec{\phi}}{dn}\right)m_\lambda \\
& \quad + \left(\frac{d^3\vec{\phi}}{dzdn^2} - U\frac{d^3\vec{\phi}}{dn^3} + 2U\mu\frac{d^2\phi}{dmdn} + 2L_{mn}\frac{d^2\vec{\phi}}{dmdn} + 2L_{nm}\frac{d^2\vec{\phi}}{dmdn}\right. \\
& \qquad \left. + 2\frac{dU}{dm}\frac{d^2\vec{\phi}}{dmdn} + 2L_{nn}\frac{d^2\vec{\phi}}{dn^2} - U\mu^2\frac{d\vec{\phi}}{dn} - \mu\frac{dU}{dm}\frac{d\vec{\phi}}{dn}\right)n_\lambda \\
& \quad + \left(\frac{DL^\gamma{}_\lambda}{Dn} + L^{3\gamma}(\Omega_{mn}m_\lambda + \Omega_{nn}n_\lambda) - L^3{}_\lambda(\Omega_{mn}m^\gamma + \Omega_{nn}n^\gamma)\right)\frac{d\vec{\phi}}{dn}n_\gamma\Bigg\}.
\end{aligned}
\tag{6.8}
$$

The last line, which is $\vec{\phi}_{,\gamma}\dot{\overline{\left\{\begin{matrix}\gamma\\ \lambda\delta\end{matrix}\right\}}}n^\delta$ with the terms in $\frac{d\vec{\phi}}{dm}$ discarded, can be written as

$$
\begin{aligned}
& \left(\frac{dL_{nm}}{dn} - \mu L_{mm} + \mu L_{nn} + L^{3n}\Omega_{mn} - L^{3m}\Omega_{nn}\right)\frac{d\vec{\phi}}{dn}m_\lambda \\
& + \left(\frac{dL_{nn}}{dn} - \mu L_{mn} - \mu L_{nm}\right)\frac{d\vec{\phi}}{dn}n_\lambda.
\end{aligned}
\tag{6.9}
$$

The two versions of $\ddot{\vec{\phi}}_{,\lambda}$, (5.5) and (6.8) as modified by (6.9), can now be compared component by component to produce the formulas for $\ddot{\phi}^{\alpha}_{|\lambda}$ and $\ddot{\phi}^{3}_{,\lambda}$. The only rule needed to produce compatibility between (5.5) and (6.8) is the expansion of the second fundamental form by (3.5) in (5.5). Again this comparison was done in 'Ricci', the result being:

$$
\begin{aligned}
\left[\ddot{\phi}^{\alpha}_{|\lambda}\right] \doteq m_\lambda \Bigg[& 2U\frac{dU}{dm}\frac{D^2\phi^\alpha}{Dn^2} + U^2\frac{D^3\phi^\alpha}{Dm\,Dn^2} - 2U\frac{D^3\phi^\alpha}{dz\,Dm\,Dn} - 2\frac{dU}{dm}\frac{D^2\phi^\alpha}{dz\,Dn} \\
& - \left(\frac{dU}{dz} - 2U^2\eta + 2UL_{mm} + UL_{nn}\right)\frac{D^2\phi^\alpha}{Dm\,Dn} + 2UL^\alpha{}_\beta\frac{D^2\phi^\beta}{Dm\,Dn} \\
& + \left(U\eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right) - \frac{d^2U}{dz\,dm} - \frac{dU}{dm}(L_{mm} + L_{nn})\right. \\
& \quad \left. + U\mu(L_{mm} - L_{nn}) - U\frac{dL_{nn}}{dn} + U\Omega_{nn}L^3{}_m - U\Omega_{mn}L^3{}_n\right)\frac{D\phi^\alpha}{Dn} \\
& + \left(2U\frac{DL^\alpha{}_\beta}{Dm} + 2\frac{dU}{dm}L^\alpha{}_\beta + 2U\Omega^\gamma_\beta m_\gamma L^{3\alpha} - 2U\Omega^\alpha_\gamma m^\gamma L^3{}_\beta\right. \\
& \quad \left. + 2U^2K(m_\beta n^\alpha - m^\alpha n_\beta)\right)\frac{D\phi^\beta}{Dn} - 2UM^\alpha\frac{d\phi^3}{dn}\Bigg] \\
+ n_\lambda \Bigg[& U^2\frac{D^3\phi^\alpha}{Dn^3} + \frac{D^3\phi^\alpha}{dz^2\,Dn} - 2U\frac{D^3\phi^\alpha}{dz\,Dn^2} - \left(\frac{dU}{dz} + 3UL_{nn}\right)\frac{D^2\phi^\alpha}{Dn^2} \\
& + 2UL^\alpha{}_\beta\frac{D^2\phi^\beta}{Dn^2} + 2L_{nn}\frac{D^2\phi^\alpha}{dz\,Dn} - 2L^\alpha{}_\beta\frac{D^2\phi^\beta}{dz\,Dn} \\
& - 3U\left(\frac{dU}{dm} + U\mu + L_{mn} + L_{nm}\right)\frac{D^2\phi^\alpha}{Dm\,Dn} \\
& + \left(U^2\mu^2 - \left(\frac{dU}{dm}\right)^2 + \left(U\mu - \frac{dU}{dm}\right)(L_{mn} + L_{nm}) + (L_{nn})^2 + \dot{L}_{nn}\right)\frac{D\phi^\alpha}{Dn} \\
& + \left(2U\frac{DL^\alpha{}_\beta}{Dn} - \dot{L}^\alpha{}_\beta + L^\alpha{}_\gamma L^\gamma{}_\beta - 2L^\alpha{}_\beta L_{nn}\right. \\
& \quad \left. + 3U\Omega^\gamma_\beta n_\gamma L^{3\alpha} - 3U\Omega^\alpha_\gamma n^\gamma L^3{}_\beta\right)\frac{D\phi^\beta}{Dn} \\
& + \left(\dot{L}^{3\alpha} - \frac{DL^{3\alpha}}{dz} + U\frac{DL^{3\alpha}}{Dn} - 3UN^\alpha + L^{3\beta}L^\alpha{}_\beta\right)\frac{d\phi^3}{dn}\Bigg],
\end{aligned}
\tag{6.10}
$$

where (3.5) and (3.10) have been used to compress. Using the facts that M^α and N^α vanish, (2.22), (4.3_3), (4.7_1), and (4.17), the above formula simplifies to

$$
\begin{aligned}
\left[\ddot{\phi}^{\alpha}_{|\lambda}\right] \doteq m_\lambda \Big\{ & 2U\frac{dU}{dm}\tilde{\tilde{\psi}}^{\alpha} + U^2\frac{D\tilde{\tilde{\psi}}^{\alpha}}{Dm} - 2U\frac{D^2\tilde{\psi}^{\alpha}}{dz\,Dm} - 2\frac{dU}{dm}\frac{D\tilde{\psi}^{\alpha}}{dz} \\
& - \left[\left(\frac{dU}{dz} - 2U^2\eta + 2UL_{mm} + UL_{nn}\right)\frac{D^2\phi^{\alpha}}{Dm\,Dn}\right] + 2U\left[L^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{Dm\,Dn}\right] \\
& + \left[\left(U\eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right) - \frac{d^2U}{dz\,dm} - \frac{dU}{dm}(L_{mm} + L_{nn})\right.\right. \\
& \quad \left.\left. + U\mu(L_{mm} - L_{nn}) - U\frac{dL_{nn}}{dn} + U\Omega_{nn}L^3{}_m - U\Omega_{mn}L^3{}_n\right)\frac{D\phi^{\alpha}}{Dn}\right] \\
& + \left[\left(2U\frac{DL^{\alpha}{}_{\beta}}{Dm} + 2\frac{dU}{dm}L^{\alpha}{}_{\beta} + 2U\Omega^{\gamma}_{\beta}m_{\gamma}L^{3\alpha} - 2U\Omega^{\alpha}_{\gamma}m^{\gamma}L^3{}_{\beta}\right.\right. \\
& \quad \left.\left. + 2U^2K(m_{\beta}n^{\alpha} - m^{\alpha}n_{\beta})\right)\frac{D\phi^{\beta}}{Dn}\right]\Big\} \\
+ n_\lambda \Big\{ & U^2\tilde{\tilde{\psi}}^{\alpha} + \frac{D^2\tilde{\psi}^{\alpha}}{dz^2} - 2U\frac{D\tilde{\tilde{\psi}}^{\alpha}}{dz} - \left[\left(\frac{dU}{dz} + 3UL_{nn}\right)\frac{D^2\phi^{\alpha}}{Dn^2}\right] \\
& + 2U\left[L^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{Dn^2}\right] + 2\left[L_{nn}\frac{D^2\phi^{\alpha}}{dz\,Dn}\right] - 2\left[L^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{dz\,Dn}\right] \\
& - 3U\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D^2\phi^{\alpha}}{Dm\,Dn}\right] \\
& + \left[\left\{(L_{mn} + L_{nm})\left(U\mu - \frac{dU}{dm}\right) - \left(\frac{dU}{dm}\right)^2 + (L_{nn})^2 + \dot{L}_{nn}\right\}\frac{D\phi^{\alpha}}{Dn}\right] \\
& + \left[\left(2U\frac{DL^{\alpha}{}_{\beta}}{Dn} - \dot{L}^{\alpha}{}_{\beta} + L^{\alpha}{}_{\gamma}L^{\gamma}{}_{\beta} - 2L^{\alpha}{}_{\beta}L_{nn}\right.\right. \\
& \quad \left.\left. + 3U\Omega^{\gamma}_{\beta}n_{\gamma}L^{3\alpha} - 3U\Omega^{\alpha}_{\gamma}n^{\gamma}L^3{}_{\beta}\right)\frac{D\phi^{\beta}}{Dn}\right]\Big\}.
\end{aligned}
\tag{6.11}
$$

With the same compression by (3.5), the other formula is

$$
\begin{aligned}
\left[\ddot{\phi}^{3}_{,\lambda}\right] \doteq m_\lambda \Big[& 2U\frac{dU}{dm}\frac{d^2\phi^3}{dn^2} + U^2\frac{d^3\phi^3}{dm\,dn^2} - 2U\frac{d^3\phi^3}{dz\,dm\,dn} - 2\frac{dU}{dm}\frac{d^2\phi^3}{dz\,dn} \\
& - \left(\frac{dU}{dz} - 2U^2\eta + 2UL_{mm} + UL_{nn}\right)\frac{d^2\phi^3}{dm\,dn} - 2UM_{\alpha}\frac{D\phi^{\alpha}}{Dn} \\
& + \left(U\eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right) - \frac{d^2U}{dz\,dm} - \frac{dU}{dm}(L_{mm} + L_{nn})\right. \\
& \quad \left. + U\mu(L_{mm} - L_{nn}) - U\frac{dL_{nn}}{dn} + U\Omega_{nn}L^3{}_m - U\Omega_{mn}L^3{}_n\right)\frac{d\phi^{\alpha}}{dn}\Big] \\
+ n_\lambda \Big[& U^2\frac{d^3\phi^3}{dn^3} + \frac{d^3\phi^3}{dz^2\,dn} - 2U\frac{d^3\phi^3}{dz\,dn^2} - \left(\frac{dU}{dz} + 3UL_{nn}\right)\frac{d^2\phi^3}{dn^2} \\
& + 2L_{nn}\frac{d^2\phi^3}{dz\,dn} - 3U\left(\frac{dU}{dm} + U\mu + L_{mn} + L_{nm}\right)\frac{d^2\phi^3}{dm\,dn} \\
& + \left(U^2\mu^2 - \left(\frac{dU}{dm}\right)^2 + \left(U\mu - \frac{dU}{dm}\right)(L_{mn} + L_{nm}) + (L_{nn})^2 + \dot{L}_{nn}\right)\frac{d\phi^3}{dn} \\
& + \left(-\dot{L}^3{}_{\alpha} + \frac{DL^3{}_{\alpha}}{dz} - U\frac{DL^3{}_{\alpha}}{Dn} + 3UN_{\alpha} + L^3{}_{\beta}L^{\beta}{}_{\alpha} + 2U\Omega^{\gamma}_{\alpha}m_{\gamma}\left(\frac{dU}{dm} + L_{nm}\right)\right)\frac{D\phi^{\alpha}}{Dn}\Big],
\end{aligned}
\tag{6.12}
$$

which simplifies using (1.7_2), (2.16), (2.23), (4.3_4), (4.7_2), and (4.18) to

$$\begin{aligned}\left[\ddot{\phi}^3_{,\lambda}\right] \doteq m_\lambda\Big\{&2U\frac{dU}{dm}\tilde{\tilde{\psi}}^3 + U^2\frac{d\tilde{\tilde{\psi}}^3}{dm} - 2U\frac{d^2\tilde{\psi}^3}{dz\,dm} - 2\frac{dU}{dm}\frac{d\tilde{\psi}^3}{dz}\\ &- \left[\left(\frac{dU}{dz} - 2U^2\eta + 2UL_{mm} + UL_{nn}\right)\frac{d^2\phi^3}{dm\,dn}\right]\\ &+ \left[\left\{U\eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right) - \frac{d^2U}{dz\,dm} - \frac{dU}{dm}(L_{mm} + L_{nn})\right.\right.\\ &\left.\left. + U\mu(L_{mm} - L_{nn}) - U\frac{dL_{nn}}{dn} + U\Omega_{nn}L^3{}_m - U\Omega_{mn}L^3{}_n\right\}\frac{d\phi^3}{dn}\right]\Big\}\\ + n_\lambda\Big\{&U^2\tilde{\tilde{\psi}}^3 + \frac{d^2\tilde{\psi}^3}{dz^2} - 2U\frac{d\tilde{\tilde{\psi}}^3}{dz} - \left[\left(\frac{dU}{dz} + 3UL_{nn}\right)\frac{d^2\phi^3}{dn^2}\right]\\ &+ 2\left[L_{nn}\frac{d^2\phi^3}{dz\,dn}\right] - 3U\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d^2\phi^3}{dm\,dn}\right]\\ &+ \left[\left\{(L_{mn} + L_{nm})\left(U\mu - \frac{dU}{dm}\right) - \left(\frac{dU}{dm}\right)^2 + (L_{nn})^2 + \dot{L}_{nn}\right\}\frac{d\phi^3}{dn}\right]\\ &+ 2U\Omega^\gamma_\alpha m_\gamma\left(\frac{dU}{dm} + L_{nm}\right)\tilde{\psi}^\alpha\Big\}.\end{aligned} \tag{6.13}$$

The third pair of formulas requires $\dddot{\vec{\phi}}$ expanded in directional derivatives. This is easily obtained by differentiation of $\ddot{\vec{\phi}}$ with respect to time using d/dz and d/dn.

$$\dddot{\vec{\phi}} = \frac{d\ddot{\vec{\phi}}}{dz} - U\frac{d\ddot{\vec{\phi}}}{dn} = \frac{d\ddot{\vec{\phi}}}{dz} - U\ddot{\vec{\phi}}_{,\lambda}n^\lambda. \tag{6.14}$$

The second term is available as the coefficient of n_λ in the expansion (7.8). The first term needs to be taken from equation (4.14) of [7],

$$\begin{aligned}\ddot{\vec{\phi}} = &\frac{d^2\vec{\phi}}{dz^2} - 2U\frac{d^2\vec{\phi}}{dz\,dn} + U^2\frac{d^2\vec{\phi}}{dn^2}\\ &- U\left(U\mu + \frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d\vec{\phi}}{dm} - \left(\frac{dU}{dz} + UL_{nn}\right)\frac{d\vec{\phi}}{dn}.\end{aligned} \tag{6.15}$$

The expression wanted is

$$\begin{aligned}\frac{d\ddot{\vec{\phi}}}{dz} = &\frac{d^3\vec{\phi}}{dz^3} - 2\frac{dU}{dz}\frac{d^2\vec{\phi}}{dz\,dn} - 2U\frac{d^3\vec{\phi}}{dz^2\,dn} + 2U\frac{dU}{dz}\frac{d^2\vec{\phi}}{dn^2} + U^2\frac{d^3\vec{\phi}}{dz\,dn^2}\\ &- \frac{d}{dz}\left\{U\left(U\mu + \frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d\vec{\phi}}{dm}\right\}\\ &- \left(\frac{d^2U}{dz^2} + \frac{dU}{dz}L_{nn} + U\frac{dL_{nn}}{dz}\right)\frac{d\vec{\phi}}{dn} - \left(\frac{dU}{dz} + UL_{nn}\right)\frac{d^2\vec{\phi}}{dz\,dn}\\ \doteq &-2U\frac{d^3\vec{\phi}}{dz^2\,dn} + 2U\frac{dU}{dz}\frac{d^2\vec{\phi}}{dn^2} + U^2\frac{d^3\vec{\phi}}{dz\,dn^2}\\ &- \left(\frac{d^2U}{dz^2} + \frac{dU}{dz}L_{nn} + U\frac{dL_{nn}}{dz}\right)\frac{d\vec{\phi}}{dn} - \left(3\frac{dU}{dz} + UL_{nn}\right)\frac{d^2\vec{\phi}}{dz\,dn}.\end{aligned} \tag{6.16}$$

The two versions (5.7) and (6.14) of $\dddot{\vec{\phi}}$ are now comparable in 'Ricci' with the derivatives of vectors all expanded in co-ordinates. The formulas produced (using (3.5) both to expand and to compress) are:

$$\begin{aligned}\left[\dddot{\phi}^{\alpha}\right] \doteq \Bigg[&-U^3\frac{D^3\phi^\alpha}{Dn^3}+3U^2\frac{D^3\phi^\alpha}{dz\,Dn^2}-3U\frac{D^3\phi^\alpha}{dz^2\,Dn}-UL_{nn}\left(2+3\frac{dU}{dz}\right)\frac{D^2\phi^\alpha}{dz\,Dn}+6UL^\alpha{}_\beta\frac{D^2\phi^\beta}{dz\,Dn}\\
&+3U\left(\frac{dU}{dz}+UL_{nn}\right)\frac{D^2\phi^\alpha}{Dn^2}-3U^2L^\alpha{}_\beta\frac{D^2\phi^\beta}{Dn^2}+3U^2\left(\frac{dU}{dm}+U\mu+L_{mn}+L_{nm}\right)\frac{D^2\phi^\alpha}{Dm\,Dn}\\
&+\left\{U\left(\frac{dU}{dm}\right)^2-\frac{D^2U}{dz^2}-U^3\mu^2+U\left(\frac{dU}{dm}-U\mu\right)(L_{mn}+L_{nm})-U(L_{nn})^2-U\dot{L}_{nn}-\frac{dUL_{nn}}{dz}\right\}\frac{D\phi^\alpha}{Dn}\\
&+3\Bigg\{\frac{dU}{dz}L^\alpha{}_\beta+U^3\Omega_{nn}\Omega^\alpha_\beta-U^3\Omega^\alpha_\gamma n^\gamma\Omega^\delta_\beta n_\delta+U^2L^3{}_\beta\Omega^\alpha_\gamma n^\gamma-U^2L^{3\alpha}\Omega^\gamma_\beta n_\gamma\\
&\quad+U\dot{L}^\alpha{}_\beta-UL^\alpha{}_\gamma L^\gamma{}_\beta+UL_{nn}L^\alpha{}_\beta\Bigg\}\frac{D\phi^\beta}{Dn}\\
&+3\Bigg\{-U\dot{L}^{3\alpha}+U\frac{DL^{3\alpha}}{dz}+U^2\frac{DL^{3\alpha}}{Dn}-U^2N^\alpha-UL^{3\beta}L^\alpha{}_\beta\\
&\quad-\left(\frac{dU}{dz}-\frac{dU}{dz}UL_{nn}+\frac{UL_{nn}}{3}\right)(L^{3\alpha}+U\Omega^\alpha_\beta n^\beta)\Bigg\}\frac{d\phi^3}{dn}\Bigg]\end{aligned} \tag{6.17}$$

and

$$\begin{aligned}\left[\dddot{\phi}^{3}\right] \doteq \Bigg[&-U^3\frac{d^3\phi^3}{dn^3}+3U^2\frac{d^3\phi^3}{dz\,dn^2}-3U\frac{d^3\phi^3}{dz^2\,dn}-UL_{nn}\left(2+3\frac{dU}{dz}\right)\frac{d^2\phi^3}{dz\,dn}\\
&+3U\left(\frac{dU}{dz}+UL_{nn}\right)\frac{d^2\phi^3}{dn^2}+3U^2\left(\frac{dU}{dm}+U\mu+L_{mn}+L_{nm}\right)\frac{d^2\phi^3}{dm\,dn}\\
&+\left\{U\left(\frac{dU}{dm}\right)^2-\frac{d^2U}{dz^2}-U^3\mu^2+U\left(\frac{dU}{dm}-U\mu\right)(L_{mn}+L_{nm})-U(L_{nn})^2-U\dot{L}_{nn}-\frac{dUL_{nn}}{dz}\right\}\frac{d\phi^3}{dn}\\
&+3\Bigg\{U\dot{L}^3{}_\alpha-U\frac{DL^3{}_\alpha}{dz}+U^2\frac{DL^3{}_\alpha}{Dn}-U^2N_\alpha-2U^2\left(\frac{dU}{dm}+L_{nm}\right)\Omega^\beta_\alpha m_\beta-UL^3{}_\beta L^\beta{}_\alpha\\
&\quad+\left(\frac{dU}{dz}-\frac{dU}{dz}UL_{nn}+\frac{UL_{nn}}{3}\right)(L^3{}_\alpha+U\Omega^\beta_\alpha n_\beta)\Bigg\}\frac{D\phi^\alpha}{Dn}\Bigg].\end{aligned} \tag{6.18}$$

These simplify using (2.22) and (2.23) respectively, (4.3), (4.7), (4.17) and (4.18) to

$$\begin{aligned}\left[\dddot{\phi}^{\alpha}\right] \doteq \Bigg[&-U^3\tilde{\tilde{\psi}}^\alpha+3U^2\frac{D\tilde{\tilde{\psi}}^\alpha}{dz}-3U\frac{D^2\tilde{\psi}^\alpha}{dz^2}-U\left(2+3\frac{dU}{dz}\right)\left[L_{nn}\frac{D^2\phi^\alpha}{dz\,Dn}\right]+6U\left[L^\alpha{}_\beta\frac{D^2\phi^\beta}{dz\,Dn}\right]\\
&+3U\left[\left(\frac{dU}{dz}+UL_{nn}\right)\frac{D^2\phi^\alpha}{Dn^2}\right]-3U^2\left[L^\alpha{}_\beta\frac{D^2\phi^\beta}{Dn^2}\right]+3U^2\left[\left(\frac{dU}{dm}+L_{mn}+L_{nm}\right)\frac{D^2\phi^\alpha}{Dm\,Dn}\right]\\
&+\left[\left\{U\left(\frac{dU}{dm}\right)^2-\frac{D^2U}{dz^2}+U\left(\frac{dU}{dm}-U\mu\right)(L_{mn}+L_{nm})-U(L_{nn})^2-U\dot{L}_{nn}-\frac{dUL_{nn}}{dz}\right\}\frac{D\phi^\alpha}{Dn}\right]\\
&+3\Bigg[\Bigg\{\frac{dU}{dz}L^\alpha{}_\beta+U^3\Omega_{nn}\Omega^\alpha_\beta-U^3\Omega^\alpha_\gamma n^\gamma\Omega^\delta_\beta n_\delta+U^2L^3{}_\beta\Omega^\alpha_\gamma n^\gamma-U^2L^{3\alpha}\Omega^\gamma_\beta n_\gamma\\
&\quad+U\dot{L}^\alpha{}_\beta-UL^\alpha{}_\gamma L^\gamma{}_\beta+UL_{nn}L^\alpha{}_\beta\Bigg\}\frac{D\phi^\beta}{Dn}\Bigg]\\
&+3\left[\left\{2U^2\frac{DL^{3\alpha}}{Dn}-\left(\frac{dU}{dz}-\frac{dU}{dz}UL_{nn}+\frac{UL_{nn}}{3}\right)(L^{3\alpha}+U\Omega^\alpha_\beta n^\beta)\right\}\frac{d\phi^3}{dn}\right]\end{aligned} \tag{6.19}$$

and

$$\begin{aligned}\left[\dddot{\phi}^3\right] \doteq & -U^3\tilde{\tilde{\tilde{\psi}}}^3 + 3U^2\frac{d\tilde{\tilde{\psi}}^3}{dz} - 3U\frac{d^2\tilde{\psi}^3}{dz^2} - U\left(2+3\frac{dU}{dz}\right)\left[L_{nn}\frac{d^2\phi^3}{dz\,dn}\right] \\ & + 3U\left[\left(\frac{dU}{dz}+UL_{nn}\right)\frac{d^2\phi^3}{dn^2}\right] + 3U^2\left[\left(\frac{dU}{dm}+L_{mn}+L_{nm}\right)\frac{d^2\phi^3}{dm\,dn}\right] \\ & + \left[\left\{U\left(\frac{dU}{dm}\right)^2 - \frac{d^2U}{dz^2} + U\left(\frac{dU}{dm}-U\mu\right)(L_{mn}+L_{nm}) - U(L_{nn})^2 - U\dot{L}_{nn} - \frac{dUL_{nn}}{dz}\right\}\frac{d\phi^3}{dn}\right] \\ & + 3\left[\left\{\left(\frac{dU}{dz} - \frac{dU}{dz}UL_{nn} + \frac{UL_{nn}}{3}\right)(L^3{}_\alpha + U\Omega^\beta_\alpha n_\beta) - 2U^2\left(\frac{dU}{dm}+L_{nm}\right)\Omega^\beta_\alpha m_\beta\right\}\frac{D\phi^\alpha}{Dn}\right].\end{aligned} \tag{6.20}$$

7 Oblique Trajectories

The usefulness of compatibility conditions with oblique trajectories, which can be chosen to be bicharacteristics of the partial differential equation being studied, has been well established. Nor is it difficult to find them. The operator $d/d\tau$ was defined in section two in terms of d/dz and d/dm. To produce compatibility conditions using $d/d\tau$ it is necessary only to replace d/dz and its iterates in the results of the previous section by their equivalents in terms of $d/d\tau$ and d/dm,

$$\frac{D\phi^\alpha}{dz} = \frac{D\phi^\alpha}{d\tau} - \gamma\frac{D\phi^\alpha}{Dm}, \tag{7.1}$$

and correspondingly for ϕ^3. When this formula is iterated, as it must be, commutation relations like those of section four of [7] are needed. Because of the vectorial commutation relation (3.22) derived above with $\doteq$, the commutativity conditions (4.1) of [7] hold here with the usual simplifying assumptions

$$\begin{aligned}\frac{D^3\phi^\alpha}{Dm\,d\tau\,Dn} &\doteq \frac{dD^2\phi^\alpha}{d\tau\,Dm\,Dn} + \left(\frac{d\gamma}{dm} - U\eta + L_{mm}\right)\frac{D^2\phi^\alpha}{Dm\,Dn}, \\ \frac{d^3\phi^3}{dm\,d\tau\,dn} &\doteq \frac{d^3\phi^3}{d\tau\,dm\,dn} + \left(\frac{d\gamma}{dm} - U\eta + L_{mm}\right)\frac{d^2\phi^3}{dm\,dn}.\end{aligned} \tag{7.2}$$

And so, analogously to (4.12) of [7], we obtain the appropriate substitutions:

$$\begin{aligned}\frac{D^3\phi^\alpha}{dz^2\,Dn} &\doteq \frac{D^3\phi^\alpha}{d\tau^2\,Dn} - 2\gamma\frac{D^3\phi^\alpha}{d\tau\,Dm\,Dn} + \gamma^2\frac{D^3\phi^\alpha}{Dm^2\,Dn} + \left(\gamma U\eta - \gamma L_{mm} - \frac{d\gamma}{d\tau}\right)\frac{D^2\phi^\alpha}{Dm\,Dn}, \\ \frac{d^3\phi^3}{dz^2\,dn} &\doteq \frac{d^3\phi^3}{d\tau^2\,dn} - 2\gamma\frac{d^3\phi^3}{d\tau\,dm\,dn} + \gamma^2\frac{d^3\phi^3}{dm^2\,dn} + \left(\gamma U\eta - \gamma L_{mm} - \frac{d\gamma}{d\tau}\right)\frac{d^2\phi^3}{dm\,dn}.\end{aligned} \tag{7.3}$$

When these formulas are applied to expressions (6.1), (6.3), (6.10), (6.12), (6.17), and (6.18), they still simplify as they did in section six, and the results are the six kinematic compatibility conditions of third order for a surface vector field under the two simplifying assumptions set out in section one, the less and the

more stringent, signaled by $\doteq$ and $\approx$:

$$
\begin{aligned}
\left[\overline{\dot{\phi}^{\alpha}_{|\beta\gamma}}\right] \doteq & \left\{\eta U\tilde{\bar{\psi}}^{\alpha} - U\frac{D^2\tilde{\psi}^{\alpha}}{Dm^2} + \left(\eta\gamma - 2\frac{dU}{dm}\right)\frac{D\tilde{\psi}^{\alpha}}{Dm} - \eta\frac{D\tilde{\psi}^{\alpha}}{d\tau}\right. \\
& \left. + \left[\left(KU + \eta^2 U - \frac{d\eta}{dz} - 2\eta L_{mm}\right)\frac{D\phi^{\alpha}}{Dn}\right]\right\} m_{\beta}m_{\gamma} \\
& + \left\{\frac{D^2\tilde{\psi}^{\alpha}}{d\tau\, Dm} - \gamma\frac{D^2\tilde{\psi}^{\alpha}}{Dm^2} - U\frac{D\tilde{\bar{\psi}}^{\alpha}}{Dm} - \frac{dU}{dm}\tilde{\bar{\psi}}^{\alpha} - \left[L^{\alpha}{}_{\delta}\frac{D^2\phi^{\delta}}{Dm\, Dn}\right]\right. \\
& \quad + \left[(L_{mm} + L_{nn} - 2\eta U)\frac{D^2\phi^{\alpha}}{Dm\, Dn}\right] - \eta\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D\phi^{\alpha}}{Dn}\right] \\
& \quad \left. + \left[(Um^{\delta}n^{\epsilon}R^{\alpha}{}_{\lambda\delta\epsilon} + \mu L^{\alpha}{}_{\lambda})\frac{D\phi^{\lambda}}{Dn}\right]\right\}(m_{\beta}n_{\gamma} + n_{\beta}m_{\gamma}) \\
& + Um^{\delta}n^{\epsilon}\left[R^{\alpha}{}_{\lambda\delta\epsilon}\frac{D\phi^{\lambda}}{Dn}\right] m_{\beta}n_{\gamma} \\
& + \left\{\frac{D\tilde{\bar{\psi}}^{\alpha}}{d\tau} - \gamma\frac{D\tilde{\bar{\psi}}^{\alpha}}{Dm} - U\tilde{\bar{\bar{\psi}}}^{\alpha} + 2\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D^2\phi^{\alpha}}{Dm\, Dn}\right]\right. \\
& \quad \left. + 2\left[L_{nn}\frac{D^2\phi^{\alpha}}{Dn^2}\right] - \left[L^{\alpha}{}_{\delta}\frac{D^2\phi^{\delta}}{Dn^2}\right] - \mu\left[L^{\alpha}{}_{\delta}\frac{D\phi^{\delta}}{Dn}\right]\right\} n_{\beta}n_{\gamma} \\
& + \left[L^{3\alpha}\frac{d\phi^3}{dn}\right](\eta m_{\beta}m_{\gamma} - \mu m_{\beta}n_{\gamma} - \eta n_{\beta}m_{\gamma} + \mu n_{\beta}n_{\gamma}) \\
\approx & \, \eta U\tilde{\bar{\psi}}^{\alpha}m_{\beta}m_{\gamma} - \left\{U\frac{D\tilde{\bar{\psi}}^{\alpha}}{Dm} + \frac{dU}{dm}\tilde{\bar{\psi}}^{\alpha}\right\}(m_{\beta}n_{\gamma} + n_{\beta}m_{\gamma}) \\
& + \left\{\frac{D\tilde{\bar{\psi}}^{\alpha}}{d\tau} - \gamma\frac{D\tilde{\bar{\psi}}^{\alpha}}{Dm} - U\tilde{\bar{\bar{\psi}}}^{\alpha} + \left[2L_{nn}\frac{D^2\phi^{\alpha}}{Dn^2} - L^{\alpha}{}_{\delta}\frac{D^2\phi^{\delta}}{Dn^2}\right]\right\} n_{\beta}n_{\gamma}
\end{aligned}
\tag{7.4}
$$

and

$$
\begin{aligned}
\left[\dot{\overline{\phi^3_{|\beta\gamma}}}\right] \doteq & \left\{\eta U\tilde{\bar{\psi}}^3 - U\frac{d^2\tilde{\psi}^3}{dm^2} + \left(\eta\gamma - 2\frac{dU}{dm}\right)\frac{d\tilde{\psi}^3}{dm} - \eta\frac{d\tilde{\psi}^3}{d\tau}\right. \\
& \left. + \left[\left(KU + \eta^2 U - \frac{d\eta}{dz} - 2\eta L_{mm}\right)\frac{d\phi^3}{dn}\right]\right\} m_\beta m_\gamma \\
& + \left\{\frac{d^2\tilde{\psi}^3}{d\tau\, dm} - \gamma\frac{d^2\tilde{\psi}^3}{dm^2} - U\frac{d\tilde{\bar{\psi}}^3}{dm} - \frac{dU}{dm}\tilde{\bar{\psi}}^3 + \left[(L_{mm} + L_{nn} - 2\eta U)\frac{d^2\phi^3}{dm\, dn}\right]\right. \\
& \left. - \eta\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d\phi^3}{dn}\right]\right\}(m_\beta n_\gamma + n_\beta m_\gamma) \\
& + \left\{\frac{d\tilde{\bar{\psi}}^3}{d\tau} - \gamma\frac{d\tilde{\bar{\psi}}^3}{dm} - U\tilde{\bar{\bar{\psi}}}^3 + 2\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d^2\phi^3}{dm\, dn}\right]\right. \\
& \left. + 2\left[L_{nn}\frac{d^2\phi^3}{dn^2}\right] - \mu\left[L^3{}_\alpha\frac{D\phi^\alpha}{Dn}\right]\right\} n_\beta n_\gamma \\
& + \left[L^3{}_\alpha\frac{D\phi^\alpha}{Dn}\right](-\eta m_\beta m_\gamma + \mu m_\beta n_\gamma + \eta n_\beta m_\gamma - \mu n_\beta n_\gamma) \\
\approx & \ \eta U\tilde{\bar{\psi}}^3 m_\beta m_\gamma - \left\{U\frac{d\tilde{\bar{\psi}}^3}{dm} + \frac{dU}{dm}\tilde{\bar{\psi}}^3\right\}(m_\beta n_\gamma + n_\beta m_\gamma) \\
& + \left\{\frac{d\tilde{\bar{\psi}}^3}{d\tau} - \gamma\frac{d\tilde{\bar{\psi}}^3}{dm} - U\tilde{\bar{\bar{\psi}}}^3 + 2\left[L_{nn}\frac{d^2\phi^3}{dn^2}\right]\right\} n_\beta n_\gamma,
\end{aligned}
\tag{7.5}
$$

$$
\begin{aligned}
\left[\ddot{\phi}^{\alpha}_{|\lambda}\right] \doteq m_\lambda \Bigg\{ & 2U\frac{dU}{dm}\tilde{\tilde{\psi}}^{\alpha} + U^2\frac{D\tilde{\tilde{\psi}}^{\alpha}}{Dm} - 2U\frac{D^2\tilde{\psi}^{\alpha}}{d\tau\, Dm} + 2U\gamma\frac{D^2\tilde{\psi}^{\alpha}}{Dm^2} - 2\frac{dU}{dm}\frac{D\tilde{\psi}^{\alpha}}{d\tau} \\
& - \left[\left(\frac{dU}{dz} - 2\gamma\frac{dU}{dm} - 2U^2\eta + 2UL_{mm} + UL_{nn}\right)\frac{D^2\phi^{\alpha}}{Dm\, Dn}\right] + 2U\left[L^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{Dm\, Dn}\right] \\
& + \Bigg[\Big\{U\eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right) - \frac{d^2U}{dz\, dm} - \frac{dU}{dm}(L_{mm} + L_{nn}) \\
& \quad + U\mu(L_{mm} - L_{nn}) - U\frac{dL_{nn}}{dn} + U\Omega_{nn}L^3{}_m - U\Omega_{mn}L^3{}_n\Big\}\frac{D\phi^{\alpha}}{Dn}\Bigg] \\
& + \Bigg[\Bigg(2U\frac{DL^{\alpha}{}_{\beta}}{Dm} + 2\frac{dU}{dm}L^{\alpha}{}_{\beta} + 2U\Omega^{\gamma}_{\beta}m_\gamma L^{3\alpha} - 2U\Omega^{\alpha}_{\gamma}m^{\gamma}L^3{}_{\beta} \\
& \quad + 2U^2K(m_\beta n^{\alpha} - m^{\alpha}n_\beta)\Bigg)\frac{D\phi^{\beta}}{Dn}\Bigg]\Bigg\} \\
& + n_\lambda\Bigg\{U^2\tilde{\tilde{\psi}}^{\alpha} + \frac{D^2\tilde{\psi}^{\alpha}}{d\tau^2} - 2\gamma\frac{D^2\tilde{\phi}^{\alpha}}{d\tau\, Dm} + \gamma^2\frac{D^2\tilde{\phi}^{\alpha}}{Dm^2} - 2U\frac{D\tilde{\tilde{\psi}}^{\alpha}}{d\tau} - \left[\left(\frac{dU}{dz} + 3UL_{nn}\right)\frac{D^2\phi^{\alpha}}{Dn^2}\right] \\
& + 2U\gamma\frac{D\tilde{\tilde{\psi}}^{\alpha}}{Dm} + 2\left[UL^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{Dn^2} + L_{nn}\frac{D^2\phi^{\alpha}}{d\tau\, Dn} - L^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{d\tau\, Dn} + \gamma L^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{Dm\, Dn}\right] \\
& - \left[\frac{d\gamma}{d\tau} + \gamma L_{mm} + 2\gamma L_{nn} - U\eta\gamma + 3U\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D^2\phi^{\alpha}}{Dm\, Dn}\right] \\
& + \left[\Big\{(L_{mn} + L_{nm})\left(U\mu - \frac{dU}{dm}\right) - \left(\frac{dU}{dm}\right)^2 + (L_{nn})^2 + \dot{L}_{nn}\Big\}\frac{D\phi^{\alpha}}{Dn}\right] \\
& + \Bigg[\Bigg(2U\frac{DL^{\alpha}{}_{\beta}}{Dn} - \dot{L}^{\alpha}{}_{\beta} + L^{\alpha}{}_{\gamma}L^{\gamma}{}_{\beta} - 2L^{\alpha}{}_{\beta}L_{nn} \\
& \quad + 3U\Omega^{\gamma}_{\beta}n_\gamma L^{3\alpha} - 3U\Omega^{\alpha}_{\gamma}n^{\gamma}L^3{}_{\beta}\Bigg)\frac{D\phi^{\beta}}{Dn}\Bigg]\Bigg\} \\
\approx m_\lambda \Bigg\{ & 2U\frac{dU}{dm}\tilde{\tilde{\psi}}^{\alpha} + U^2\frac{D\tilde{\tilde{\psi}}^{\alpha}}{Dm}\Bigg\} \\
& + n_\lambda\Bigg\{U^2\tilde{\tilde{\psi}}^{\alpha} - 2U\frac{D\tilde{\tilde{\psi}}^{\alpha}}{d\tau} + 2U\gamma\frac{D\tilde{\tilde{\psi}}^{\alpha}}{Dm} \\
& + \left[2UL^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{Dn^2} - \left(\frac{dU}{dz} + 3UL_{nn}\right)\frac{D^2\phi^{\alpha}}{Dn^2}\right]\Bigg\}
\end{aligned}
\tag{7.6}
$$

and

$$
\begin{aligned}
\left[\ddot{\phi}^{3}_{,\lambda}\right] \doteq m_\lambda \Bigg\{ & 2U\frac{dU}{dm}\tilde{\tilde{\psi}}^3 + U^2\frac{d\tilde{\tilde{\psi}}^3}{dm} - 2U\frac{d^2\tilde{\psi}^3}{d\tau\, dm} + 2U\gamma\frac{d^2\tilde{\psi}^3}{dm^2} - 2\frac{dU}{dm}\frac{d\tilde{\psi}^3}{d\tau} \\
& - \left[\left(\frac{dU}{dz} - 2\gamma\frac{dU}{dm} - 2U^2\eta + 2UL_{mm} + UL_{nn}\right)\frac{d^2\phi^3}{dm\, dn}\right] \\
& + \Bigg[\left\{U\eta\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right) - \frac{d^2U}{dz\, dm} - \frac{dU}{dm}(L_{mm} + L_{nn})\right. \\
& \left. + U\mu(L_{mm} - L_{nn}) - U\frac{dL_{nn}}{dn} + U\Omega_{nn}L^3{}_m - U\Omega_{mn}L^3{}_n\right\}\frac{d\phi^3}{dn}\Bigg]\Bigg\} \\
& + n_\lambda\Bigg\{U^2\tilde{\tilde{\psi}}^3 + \frac{d^2\tilde{\psi}^3}{d\tau^2} - 2U\frac{d\tilde{\tilde{\psi}}^3}{d\tau} + 2U\gamma\frac{d\tilde{\tilde{\psi}}^3}{dm} - \left[\left(\frac{dU}{dz} + 3UL_{nn}\right)\frac{d^2\phi^3}{dn^2}\right] \\
& - 2\gamma\frac{d^2\tilde{\psi}^3}{d\tau\, dm} + \gamma^2\frac{d^2\tilde{\psi}^3}{dm^2} + 2\left[L_{nn}\frac{d^2\phi^3}{d\tau\, dn}\right] \\
& - \left[\frac{d\gamma}{d\tau} - U\gamma\eta + \gamma L_{mm} + 2\gamma L_{nn} + 3U\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d^2\phi^3}{dm\, dn}\right] \\
& + \left[\left\{(L_{mn} + L_{nm})\left(U\mu - \frac{dU}{dm}\right) - \left(\frac{dU}{dm}\right)^2 + (L_{nn})^2 + \dot{L}_{nn}\right\}\frac{d\phi^3}{dn}\right] \\
& + 2U\Omega^\gamma_\alpha m_\gamma\left(\frac{dU}{dm} + L_{nm}\right)\tilde{\psi}^\alpha\Bigg\} \\
\approx\ & m_\lambda\left\{2U\frac{dU}{dm}\tilde{\tilde{\psi}}^3 + U^2\frac{d\tilde{\tilde{\psi}}^3}{dm}\right\} \\
& + n_\lambda\left\{U^2\tilde{\tilde{\psi}}^3 - 2U\frac{d\tilde{\tilde{\psi}}^3}{d\tau} + 2U\gamma\frac{d\tilde{\tilde{\psi}}^3}{dm} - \left[\left(\frac{dU}{dz} + 3UL_{nn}\right)\frac{d^2\phi^3}{dn^2}\right]\right\},
\end{aligned}
\tag{7.7}
$$

and

$$\begin{aligned}
\left[\dddot{\phi}^{\alpha}\right] \doteq & -U^3\dddot{\tilde{\psi}}^{\alpha} + 3U^2\frac{D\ddot{\tilde{\psi}}^{\alpha}}{d\tau} - 3U\frac{D^2\dot{\tilde{\psi}}^{\alpha}}{d\tau^2} - U\left(2+3\frac{dU}{dz}\right)\left[L_{nn}\frac{D^2\phi^{\alpha}}{d\tau\,Dn} - \gamma L_{nn}\frac{D^2\phi^{\alpha}}{Dm\,Dn}\right] + 6U\left[L^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{d\tau\,Dn}\right] \\
& + 6U\gamma\frac{D^2\dot{\tilde{\psi}}^{\alpha}}{d\tau\,Dm} - 3U^2\gamma\frac{D\ddot{\tilde{\psi}}^{\alpha}}{Dm} - 3U\gamma^2\frac{D^2\dot{\tilde{\psi}}^{\alpha}}{Dm^2} + 3U\left(\frac{d\gamma}{d\tau} + \gamma L_{mm} - U\gamma\eta\right)\frac{D\dot{\tilde{\psi}}^{\alpha}}{Dm} - 6U\gamma\left[L^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{Dm\,Dn}\right] \\
& + 3U\left[\left(\frac{dU}{dz} + UL_{nn}\right)\frac{D^2\phi^{\alpha}}{Dn^2}\right] - 3U^2\left[L^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{Dn^2}\right] + 3U^2\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{D^2\phi^{\alpha}}{Dm\,Dn}\right] \\
& + \left[\left\{U\left(\frac{dU}{dm}\right)^2 - \frac{D^2U}{dz^2} + U\left(\frac{dU}{dm} - U\mu\right)(L_{mn}+L_{nm}) - U(L_{nn})^2 - U\dot{L}_{nn} - \frac{dU\,L_{nn}}{dz}\right\}\frac{D\phi^{\alpha}}{Dn}\right] \\
& + 3\Bigg[\Bigg\{\frac{dU}{dz}L^{\alpha}{}_{\beta} + U^3\Omega_{nn}\Omega^{\alpha}_{\beta} - U^3\Omega^{\alpha}_{\gamma}n^{\gamma}\Omega^{\delta}_{\beta}n_{\delta} + U^2L^3{}_{\beta}\Omega^{\alpha}_{\gamma}n^{\gamma} - U^2L^{3\alpha}\Omega^{\gamma}_{\beta}n_{\gamma} \\
& \quad + U\dot{L}^{\alpha}{}_{\beta} - UL^{\alpha}{}_{\gamma}L^{\gamma}{}_{\beta} + UL_{nn}L^{\alpha}{}_{\beta}\Bigg\}\frac{D\phi^{\beta}}{Dn}\Bigg] \\
& + 3\left[\left\{2U^2\frac{DL^{3\alpha}}{Dn} - \left(\frac{dU}{dz} - \frac{dU}{dz}UL_{nn} + \frac{UL_{nn}}{3}\right)(L^{3\alpha} + U\Omega^{\alpha}_{\beta}n^{\beta})\right\}\frac{d\phi^3}{dn}\right] \\
\approx & -U^3\dddot{\tilde{\psi}}^{\alpha} + 3U^2\frac{D\ddot{\tilde{\psi}}^{\alpha}}{d\tau} - 3U^2\gamma\frac{D\ddot{\tilde{\psi}}^{\alpha}}{Dm} \\
& + 3U\left[\left(\frac{dU}{dz} + UL_{nn}\right)\frac{D^2\phi^{\alpha}}{Dn^2} - UL^{\alpha}{}_{\beta}\frac{D^2\phi^{\beta}}{Dn^2}\right]
\end{aligned} \tag{7.8}$$

and

$$\begin{aligned}
\left[\dddot{\phi}^{3}\right] \doteq & -U^3\dddot{\tilde{\psi}}^{3} + 3U^2\frac{d\ddot{\tilde{\psi}}^{3}}{d\tau} - 3U^2\gamma\frac{d\ddot{\tilde{\psi}}^{3}}{dm} - 3U\frac{d^2\tilde{\psi}^{3}}{d\tau^2} + 6U\gamma\frac{d^2\tilde{\psi}^{3}}{d\tau\,dm} - 3U\gamma^2\frac{d^2\tilde{\psi}^{3}}{dm^2} \\
& + 3U\left(\frac{d\gamma}{d\tau} + \gamma L_{mm} - U\gamma\eta\right)\frac{d\tilde{\psi}^{3}}{dm} - U\left(2+3\frac{dU}{dz}\right)\left[L_{nn}\frac{d^2\phi^{3}}{d\tau\,dn} - \gamma L_{nn}\frac{d^2\phi^{3}}{dm\,dn}\right] \\
& + 3U\left[\left(\frac{dU}{dz} + UL_{nn}\right)\frac{d^2\phi^{3}}{dn^2}\right] + 3U^2\left[\left(\frac{dU}{dm} + L_{mn} + L_{nm}\right)\frac{d^2\phi^{3}}{dm\,dn}\right] \\
& + \left[\left\{U\left(\frac{dU}{dm}\right)^2 - \frac{d^2U}{dz^2} + U\left(\frac{dU}{dm} - U\mu\right)(L_{mn}+L_{nm}) - U(L_{nn})^2 - U\dot{L}_{nn} - \frac{dU\,L_{nn}}{dz}\right\}\frac{d\phi^{3}}{dn}\right] \\
& + 3\left[\left\{\left(\frac{dU}{dz} - \frac{dU}{dz}UL_{nn} + \frac{UL_{nn}}{3}\right)(L^3{}_{\alpha} + U\Omega^{\beta}_{\alpha}n_{\beta}) - 2U^2\left(\frac{dU}{dm} + L_{nm}\right)\Omega^{\beta}_{\alpha}m_{\beta}\right\}\frac{D\phi^{\alpha}}{Dn}\right] \\
\approx & -U^3\dddot{\tilde{\psi}}^{3} + 3U^2\frac{d\ddot{\tilde{\psi}}^{3}}{d\tau} - 3U^2\gamma\frac{d\ddot{\tilde{\psi}}^{3}}{dm} + 3U\left[\left(\frac{dU}{dz} + UL_{nn}\right)\frac{d^2\phi^{3}}{dn^2}\right].
\end{aligned} \tag{7.9}$$

When $\vec{L} = 0$, these six formulas reduce to those of [1].

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