

# Meanings in Ordinary Language and in Mathematics

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In a previous paper in this journal,<sup>1</sup> I decried the lack of discussion of meaning in mathematics. It is hardly fair to do so and then to do nothing about it. Accordingly, I offer the following contribution to such a discussion. It is in three parts. The first part attempts to remedy the difficulty that a discussion of meaning in mathematics normally labors under, namely the lack of a way to talk about even ordinary-language meaning. So I supply a sketch of how ordinary meanings of the simplest fact-conveying kind can be discussed. In Part 2, I apply this way of talking to a discussion of the non-empty topic of mathematical meanings, intending most importantly to convey the non-emptiness. I distinguish between two different cases that I argue against, that mathematics is not meaningful in the terms that I am using and that mathematics cannot be made meaningful in any terms. Then in Part 3 I take up further a single point of comparison, mentioned in Part 2, between ordinary and mathematical meanings, specifically how they arise. I hold that they are not as different as they may at first appear.

As I argue in Part 2 of this essay, the subject of meanings is important for mathematics. It is of sufficiently general importance to make comments on it of potential interest to almost anyone. Meaning in mathematics is dependent on meaning in general, and that is my excuse for beginning with the latter. Some of Part 1 is here chiefly because it is needed in Part 2 on meanings in mathematics, but nothing is here solely for that reason because this part is intended to be of independent interest.

## Part 1. Meanings in ordinary language

### Introduction

The meanings that are referred to are not as general as they might be (but see the last paragraph of the addendum). In particular, I intend to discuss the sorts of meanings that are communicated from person to person by speech, writing, and diagrams, not such larger matters as the meaning of life. It is necessary to point out this distinction and elaborate on it slightly because much of what follows is dependent on the work of Mark Johnson, who does not make this distinction strictly. Johnson may be right, but it seems to me that in a phrase like the meaning of life,

meaning is being used metaphorically for something that I prefer to call significance. A human life has a significance even where it does not obviously have a message for anyone. Has it ever such a message? It is said that some lives, the classical examples being the lives of saints, do have the characteristic of being message bearers. I am inclined to think rather that their lives and the lives of others that are thought to be exemplary have a significance that can better be thought of in other terms than messages. To explain these other terms I need a distinction that has been around since Plato and is made very clearly by Michael Polanyi in *Personal knowledge*:<sup>2</sup> the distinction between "knowledge that" and "knowledge how". "Knowledge that", factual knowledge, is commonly communicated by instruction, but "knowledge how", skill, is learned by doing, from example, and only peripherally from instruction. Exemplary lives may teach one how to live, but they do not have the kind of message that could just as well have been written on a paper in bottle and washed up on the seashore. It is the meanings of such messages within or without bottles that are the subject of this essay.

The starting point of my consideration of message meanings is that a message is a representation of what a person means to express by writing or saying that message. The meaning of what is said, like the choice of what is spoken of, is a person's before it is a document's. When the document is read by another person, an interpretation will be placed upon the document that the reader thinks is the writer's meaning or the document's meaning. There are a number of ways in which the above typical situation may fail to be the case: the writer may have meant something but failed to express it, or the writer may not have intended to mean anything. It is by no means axiomatic that a document should mean something. It is a fundamental error to treat documents as though they were objects of natural history to be understood on their own merits without regard to their personal source. Equally, it is a tenet of several major religions that even the objects of natural history can be better understood in terms of a personal source. Be that as it may, all known documents have been produced by some human or by machinery following some ultimately human instructions. The dominant metaphor for this process is that the medium carries, like a container, the meaning that the speaker intends to transmit to the hearer. The treatment of texts as though they were not media of human communication has led, not only in mathematics but also in general, to the strange doctrine that is being called objectivism. The combatting of objectivism is a main aim of both Johnson's book *The body in the mind*<sup>3</sup> and George Lakoff's book *Women, fire, and dangerous things*,<sup>4</sup> where the job is done philosophically and scientifically (linguistics) respectively, and very little of their polemic will be reproduced here. Meaning, in objectivism, is a mind-independent relation between sentences and mind-independent states of affairs. I do not see how one could reasonably deny relations among sentences and objective reality. For instance a sentence can refer to some reality and can describe truly some reality, but both of these relations are mediated by the

person making the sentence, the person receiving the sentence, and the occasion. I am fully in sympathy with a lack of mind independence.

I intend to avoid the problem of what, with any degree of objectivity, is *the* meaning of a document or of a written sentence for a while, writing first about meaning as it is intended or interpreted by just one person. I shall return to it, as it is of considerable importance. It is not, however, the place to begin. One can note that before the comparatively recent invention of writing, sentences were exclusively spoken by persons to persons. The problem of documents is a derivative problem, and the problem at the personal level must have its solution independently of the documentary problem.

Personal meaning brings with it a difficulty that I am not intending to deal with now or later, and that is the large variety of sorts of meaning that can be identified in semantics to be borne by a speech. There are emotional charges and performative meanings, and a lot of others, all of which I intend to avoid, treating only what is sometimes called *descriptive meaning*. Except that the content can be fictional or lying, one might call it factual meaning.

## Metaphor

The terms in which Johnson explains meanings are those of figurative language. Anyone wishing to follow up what I have written here will be forced to consult Johnson's fine book and in doing so will encounter a difficulty that all writers on figurative language throw up against their readers. Not without some reason, Johnson calls figurative language in general by the name metaphor, which has, as well as this very general signification, the meaning of one of the many varieties of figurative language identified by students of rhetoric and poetry. Having given the word a much wider meaning than it has in the hands of many writers, Johnson is at pains to show that their idea of metaphor is much narrower than his own, that students of rhetoric and poetry mean something narrower than he does. This is obvious but confusing. The simplest and narrowest meaning of metaphor is a decorative device used by poets and well known by all. It is easy to show that it is more than decorative, and actually conveys meaning. On the other hand, what Johnson wants to say is that all meaning is in some sense figuratively based, and for this basis he wants the broadest notion of figurative language that he can have. I have been trying in this paragraph to disambiguate the term metaphor by calling the poetic device metaphor and the general phenomenon figurative language.

It is not a new idea of Johnson's that meaningful language is at bottom figurative. The oldest source of the idea mentioned by I.A. Richards in his *Philosophy of rhetoric*<sup>5</sup> is Shelley, who is quoted (no reference) as follows:

Language is vitally metaphorical; that is, it marks the before unapprehended relations of things and perpetuates their apprehension, until words, which represent them, become, through time, signs for portions or classes of thought instead of pictures of integral thoughts: and then, if no new poets should arise to create afresh the associations which have been thus disorganized, language will be dead to all the nobler purposes of human intercourse.

I do not pretend to know exactly what Shelley was driving at, but he pretty clearly has at least two ideas, both of which one can meet with elsewhere. One, that language is at bottom figurative. Two, that poets are indispensable to the creation of language. To take them out of order, I want to remark that, while Shelley had an obvious metaphorical axe to grind, he may have been right in his day. For older languages than English, for instance, Sanskrit and classical Greek, he was probably right. This idea is that words are given their initial prosaic meanings in everyday unmemorable speech, that their meanings are made memorable as well as expanded and refined by poets, and that they then come back into prose use with their expanded and enriched meanings. This idea goes back at least to Shakespeare, who wrote

...as imagination bodies forth  
The forms of things unknown, the poet's pen  
Turns them to shapes, and gives to airy nothing  
A local habitation and a name.<sup>6</sup>

At present I think that (in my interpretation) Shelley is no longer correct. Poetry is above all else memorable; it bears repeating. Some of its devices help make it possible to remember (in order to repeat it), and some help make it worth remembering. The best do both. Remembering is no longer a problem on account of literacy (not to mention recordings), and so it is no longer necessary to make something easy to remember or worth remembering in order to ensure its repetition. If repetition is the key, as I suspect it is, then the forgers of words for the present day are in a small way scientists, but in a big way the authors of the print and electronic media and in particular advertising copywriters. This has had an obviously negative effect on modes of expression (the mindless redundancy of the NDP party and the HIV virus), and presumably also a negative effect on ideas expressed (happiness is what you buy). Even on a properly literary level, the poetic monopoly has been broken by such as Nabokov, with his gift to us of nymphets.<sup>7</sup> So much for the indispensability of poets. To go back to the idea that language is at bottom figurative, I have no quarrel with this idea, which I have seen expressed in many places. It is a notion of such vast generality that even whole books on the subject of metaphor in the broad sense tend to discuss metaphor's being fundamental to only particular departments of thought. Leatherdale<sup>8</sup> and Soskice<sup>9</sup> spring to mind. Perhaps the fullest working out of what the idea might mean is Pepper's

interpretation of past metaphysics as being the discovery, elaboration, and evidential support of four *World hypotheses*<sup>10</sup> based on four "root metaphors", single metaphors of such importance that they can be applied to the world as a whole with some considerable adequacy, and of his own contribution of a world hypothesis based on what he considers to be a new root metaphor.<sup>11</sup> But again, metaphysics, while in one sense all-encompassing, is a particular department of thought. It has been left to Johnson, in his former work *Metaphors we live by*<sup>12</sup> and in the book already mentioned, to explain how it comes about that figurative language can be fundamental to all language.

There are two ways in which comparison, and so metaphor in the broadest sense, comes inevitably into the expression of meaning; one is mentioning things and relations, or categorization, and the other is actually saying something. When one uses a category word to describe a thing, one is likening the thing to other things that one would apply the same word to. Any study of this process must consider how humans use categories in practice. This Lakoff has done in his *Women, fire, and dangerous things*. When one says anything more sophisticated than pointing to a tree and saying "oak", one is doing more than categorizing, and many examples of how figures are presupposed in more complex common speech – virtually all speech – were given by Lakoff and Johnson in *Metaphors we live by*. The idea that explanation will inevitably be done in terms of some root metaphor has now spread beyond linguistics, for example to psychology.<sup>13</sup>

### Assimilation and accommodation

A paper is not the place to reproduce two books, and so I am going to leave the explanation of the Lakoff and Johnson material aside and discuss the step above that of personal meanings. I am going to do this in terms that are my own, but influenced by my recent reading of Pepper. I think of what a word applies to, whether thing or relation, as being an element of a class of my own creation, which I call an assimilation class because it is the class of what I assimilate to the other members of the class. Assimilation therefore applies to all members of this class: I assimilate them into such a class largely on a basis of family resemblance.<sup>14</sup> And assimilation also applies to a single member of the class when I come upon it. Now the first assimilation is not something that I actually do, for instance to all oak trees; it is merely a disposition that would be realized in each separate instance of my encountering an oak tree and so categorizing it.

At a fairly early stage in life one knows that some things are called by one's language's name for those things; one has formed one's initial personal class of such things, Russell's idiosyncratic particulars. Despite the differences that manifestly exist among those things, one assimilates them into a common kind of thing. Since one

does this first for natural kinds like flower and nut and dog, this is not difficult, but it is one's own intellectual activity. Then as one learns more of dogs and meets breeds of dogs that one did not know at first, one does together two things; one assimilates different looking dogs to one's personal class of dogs and one accommodates one's personal class of dogs to the new additions. These correlative processes of assimilation and accommodation are the cornerstones of the genetic epistemology of Jean Piaget.<sup>15</sup> Throughout the whole of one's life, one's use of any category is subject to these processes until and unless one develops a thoroughly closed mind. Not to admit the possibility of novel assimilations and the correlative accommodations is to claim to know all about the classes about which one makes such a claim. Such a claim is immodest and almost certainly wrong. To illustrate the way in which assimilation and accommodation are, as I put it, correlative, I turn to club membership. When a member joins a small organization, the kind in which every member makes a difference, the organization assimilates the new member, and the new member is accommodated by the organization. Accommodation is what happens to the organization, and assimilation is what happens to the member. Now obviously in a big organization there is no noticeable accommodation by the organization when it assimilates a new member; it just assigns a membership number and adds a name to a list. It is similar with the personal assimilation classes; one's assimilation class corresponding to the expression "golf ball" is not noticeably changed by seeing each white golf ball (accommodation nil), but recently, when I saw my first red golf ball, my golf-ball class was noticeably enlarged by the accommodation to colored golf balls.

An example of assimilation that normally does not involve accommodation is the use of a map to find one's way through territory that one does not know. A good reader of maps assimilates physical surroundings as they are met to the neighborhoods on the map and traces on the map movement in the terrain. This can, incidentally, be a distinctively pleasurable process; assimilation is one of the simple pleasures. Typically there is no accommodation, and what would accommodation be? It would be marking on the map features that are not marked, a new road or a new dam, or the erasure of a torn-up railway line. Accommodation is work; it is less fun than assimilation, a fact that is borne out in human cognitive behavior again and again. It is important to stress that assimilation is something that can be done regardless of the objective similarities and dissimilarities between things assimilated. It is something that we do, not something thrust upon us by our surroundings. In the example of a map, all have had the experience of assimilating their actual physical location and a place on a map that does not correspond to it in the correct interpretation of the map.

An example of accommodation that is not very hard work, in fact it is so easy that it might not be called accommodation except that it does modify one's assimilation classes, is accommodation to standard examples and away from

unmemorable exceptions. In the natural course of events, less memorable exceptional members of an assimilation class will simply be forgotten, and the class will be reduced by the elimination of former potential members whose territory was staked out by the forgotten exceptions. Because one sees the unexceptional members more frequently than exceptional members, one's recollection of their membership is reinforced, and that of the exceptional members is not. This is at least part of the explanation in these terms for why assimilation classes frequently have special prototype or exemplar members. Lakoff<sup>16</sup> has summarized a substantial literature on this aspect of classes. The resemblance of members to the exemplars is of course only the family resemblance of Wittgenstein, not the simpler sorts of resemblance studied, for instance, by some psychologists.<sup>17</sup>

There are two things to emphasize about these classes, one is that each of them is my personal creation, the result of assimilating successive examples of oak trees and learning about oak trees in other ways and accommodating my notion of oak tree to my successive assimilations and information. This process of learning has nothing to do with fuzzy sets.<sup>18</sup> An example that comes to mind is "mother", which at first means one person and then is accommodated to the inclusion of her mother than then other mothers. In a narrow sense, then, my notion of what "mother" means is not idiosyncratic, but in a broader sense, what the word "mother" means to me remains colored by my relation to my mother.

The second thing to emphasize is a result of the first, namely that my classes are not necessarily the same as the classes of others, because their classes are the results of their somewhat different experiences.<sup>19</sup> One of the ways that we coordinate the variety of things that fall into assimilation classes and coordinate the assimilation classes of the variety of persons is the use of prototypes.<sup>20</sup> There will be a lot of overlap among the assimilation classes of various observers. The process of assimilation is again used in the institutionalization<sup>21</sup> of word meanings among the speakers of the "same" language. To begin with, who I mean by the speakers of the "same" language are the persons whose personal languages I assimilate. (The use of "same" is more often a signal that assimilation has taken place than an indication of identity.) Among the speakers, then, of what I regard as the "same" language there is substantial agreement about the meanings of the words of their common language. It is therefore possible for me to assimilate those persons' meanings for the words of "the" language and create a single language. (There is, incidentally, a lower level of assimilation at which the different ways the different speakers speak each word are assimilated into classes that represent "single" words; likewise the different ways that different writers write "the same" mathematical symbol are interpreted as "the same" symbol. These classes are sometimes erroneously called equivalence classes.<sup>22</sup>) Each speaker of a language, therefore, has a personal version of the institutionalized language. The meanings that one attributes to the words of that language can be quite wide of the mark or not. How useful one's version of a language is depends,

like one's other assimilations, on how closely one's assimilation fits<sup>23</sup> the way the part of the world, which it attempts to organize, actually is. Unless one is a Platonist of some sort, however, there are no official classes to compare one's assimilation classes with, because one does not have official classes; and even the Platonist has no access to them.<sup>24</sup> Surely no one will deny me the right to my own class of golf balls, which until recently were all white? But what right have I to regard my class of golf balls – or Chippendale chairs – as the same as everyone else's? None, I freely admit, but I have no alternative to doing this if I am going to use the notion of golf ball at all. I have no access to anyone else's class of golf balls, and if I take seriously their verbal descriptions of golf balls then I am accommodating my class to theirs. This is of course something that I do, acquiring Russell's "knowledge by description". But all my doing of this alters only my own classes; I get on with life by assuming that my classes are adequate representations of the classes that are common to speakers of English. Intersubjectivity, writes Rommetveit<sup>25</sup> is "inconceivable without naive, reciprocal faith in a shared experiential world... *Intersubjectivity must in some sense be taken for granted in order to be attained.*"

When I speak to a person, I use the words of our common language in a fairly standard way, but I take some account of my expectations that the listener may have (and in particular may share with me) some peculiarities in assimilation classes. For instance, I can call rutabagas "turnips" within the bosom of my family because I know that the easier word will convey the correct meaning. When I write anything but a personal letter, where the same expectations would apply, I have to take more account of the institutionalized meanings of the words I use because I assume that my writing will be read by persons having nothing in particular in common but being speakers of English. What happens when such a person reads my sentence? The steps, according to Pepper<sup>26</sup> are that the reader reads the printed tokens, perceives the sentence as words connected in grammatical form, understands the sentence in terms of his understanding of the institutionalized meanings of the words. If the reader is interested in verifying the sentence, then it becomes necessary to do so in terms of the reader's own assimilation classes. One cannot possibly use the institutionalized meanings for anything but the construction and interpretation of texts. The meaning of my sentence for the reader, if the reader is to do anything about it, is entirely in terms of the reader's interpretation in terms of the reader's assimilation classes.<sup>27</sup> The communication of something originally thought of in terms of a writer's assimilation classes to a reader is only rather tenuously represented in terms of the usual root metaphor of a container with the meaning of the message conveyed within it. The meaning that "arrives" is created by the reader on the basis of the reader's experience; only by good luck is anything moved from writer to reader. It is in principle impossible for what is moved to be unchanged; there is no basis for an identity between what is sent and what is received. The most that is even arguable for is reasonableness of assimilation. When our personal classes are not sufficiently similar, then we do hopelessly misunderstand one another. For



public words there is no ultimate basis in public ostension, and public sentences have meaning only to persons, their writers/speakers and readers/hearers, based on their personal assimilation classes.

I should emphasize again that one's assimilation classes are very much a matter of personal decision. They are not dictated by circumstance, and their usefulness depends on how they are deployed, not entirely on how accurately they "mirror" the external world. The class boundaries are not in the external world; indeed, the classes themselves are imaginative projections onto the external world.<sup>28</sup> An example is the famous paradox of the heap (sorites). A heap is something piled up but not neatly. Normally, the removal of one of the heaped objects leaves the heap still a heap. On that basis, one can remove the heaped objects one by one, keeping the heap as such, but eventually coming to a heap of one, which no one would regard as a heap. The difficulty here is that calling something a heap is one's choice; it does not reflect something binding in the pile. It is not the case that removing one object from a heap of objects leaves the heap a heap. There must be indefinitely (not infinitely) many objects. With small objects the number of them needs to be so large that one would not notice the removal of one. With big objects like squashed automobile bodies, I can imagine calling three – if not neatly stacked – a heap, but not two.

### **Definition**

Aristotle pointed out that we cannot define everything. This is logically sound, but in fact dictionaries do define everything by the obvious device of circular definitions. One can play the game Vish by looking up successive words in a dictionary until one gets back to the first word one looked up, a vicious circle. In ordinary language, definition is in terms of assimilation classes other than those named by the term defined. It is not allowed to have a vicious point (circle with zero circumference). So Aristotle was wrong for ordinary language because he spoke of rigorous definition, which is not possible in ordinary language.

### **Reference**

The relation between meaning and reference needs to be discussed because it has led to important confusions.<sup>29</sup> As I remarked at the beginning, the choice of what is spoken about is the speaker's just like the meaning to be indicated. I think that there can be no doubt that we learn the meanings of words in childhood primarily through ostensive definitions. Our acquaintance with these idiosyncratic particulars (Russell's term) allows us to form our assimilation classes. Further acquaintance

with particulars, combined with knowledge gained in verbal form, allows us to accommodate our classes to the new particulars and the new information so that our classes approximate more closely the standard classes, that is the classes that are reflected in dictionaries and by educated speakers. These classes, one should recall, are not well defined sets by any means; they are dispositions to place things we have or have not before met into classes of things we already know about. As we grow up, these dispositions typically produce some strange classifications. Gradually, our classes become more or less standard, but from the beginning we are prepared to put the things we meet into our classes. "Men with beards are fathers" we grow out of, but "humans with beards are men" we stick with. We are by no means constrained to classify only things that actually fit our categories; it is rare for an adult or even for a child to say "I have absolutely no idea what that was", and even to say so little rather than "what happened?" classes "that" as a thing rather than an event. And correspondingly in the opposite direction, we are able to understand combinations of words as descriptions of things and expressions of ideas of which we formerly had no idea whatever. As Russell wrote, "Fiction, history, and all giving of information depend upon this property of language."<sup>30</sup> He might have added that a great deal of the power of our imaginations, outside specifically non-verbal areas like the pictorial and musical,<sup>31</sup> is also based on this property of language. He would not (so far as I know) have gone further, although I shall: since our imaginations are needed in the construction of our experience, even our experience and all having of information depend on this property of language. It is of some importance that the meanings of words – represented for me by assimilation classes, but however represented – determine their reference, not the reverse. The things to which I make a word refer are determined by my assimilation classes; it is quite impossible for the things to which a word in any sense "properly" refers to determine what a word shall mean when I use it. Logically all that can determine that for me is what I think, and that is represented by my assimilation classes. The better educated I am, the more closely my classes conform to the proper sets, supposing these to exist, and so the easier it is to make the mistake of thinking that the fictitious sets behind the making of dictionaries determine what words mean to me and what I mean by words. But it is not so because it cannot be so.<sup>32</sup>

A peculiarity of the basis of reference in what persons mean is that speakers have a freedom of reference that a disembodied text does not have. A speech act can be made by a speaker to refer in ways that the speech itself, if encountered in written form, would not. Donellan<sup>33</sup> pointed out that a speaker saying, "The man over there with the champagne in his glass is happy", provided that whom he refers to is sufficiently obvious, is referring to the contents of the man's glass as champagne regardless of what the contents really are. This illustrates the general point that with sufficient extra-linguistic force, almost any word can be made to refer to almost anything at all. One's intention, provided it is clear enough, overrides the meaning of words in their referential function so that the meaning of one's words are

accommodated to the referent that has been specified otherwise. A word can be made to refer effortlessly to items in the common assimilation class corresponding to that word, but with effort can be made to refer well outside that class. Greater force can make champagne refer to ginger ale or water. Some kinds of metaphor depend on this force. When a word is made to refer to something to which it would not normally be applied, our usual reaction is to reject its literal application and seek for a metaphorical meaning that will facilitate the forced reference. My favorite example of this is the quotation, by a person that has eaten too many potato chips, of Macbeth's line, "I have supped full with horrors." In the play, on the common assumption that Macbeth is blighted rather than bloated, the supping is metaphorical and the horrors are literal, but in everyday life we hesitate to take "horrors" literally and instead apply it, with a metaphorical meaning, to the potato chips; this allows us to reverse the process applied to "supping" in the play and shift it back to its literal meaning.

I have left until writing of reference to emphasize that in deciding what do make of the meaning of a speech – still more of a text – and deciding what the speech – still more a text – is about, one is guessing. At best this guessing is well educated, confident, and correct, or correct in the reference and as correct as possible in the meaning. Probably the context where there is the least doubt is the legal, where centuries of talented effort have been devoted to the writing of texts that are unambiguous as to content and certain as to reference. And even here the prevalence of resort to the courts and of disagreement among courts upon appeals indicates that when the stakes are higher than in daily chit-chat a lack of clarity can be found. The reason that we achieve the clarity that we do is largely that we do not bother to investigate the potential ambiguities of meaning and vagueness of reference. While I have no interest in interpretations' often being wrong, I must insist that they are fundamentally guesswork.

## **Part 2. Meanings in mathematics**

### **Introduction**

At the beginning of Part 1, I mentioned the distinction between "knowledge how" and "knowledge that", indicating that the meanings, which I was going to discuss there, were the meanings by which are conveyed "knowledge that", "knowledge how" being mostly shown by example and learned by experience. The examples by which general skills are acquired are not without meaning, or as I prefer to call it significance. Nor are examples in mathematics insignificant in the acquisition of mathematical skills. In a different way, mathematics can have significance,<sup>34</sup> its beauty being for some a theophany. But the main target of this essay is the noxious notion that written or

spoken mathematics is meaningless,<sup>35</sup> and the main kind of meaning, which I wish to attribute to mathematical expressions, is the kind that conveys "knowledge that", for instance that  $2 + 2 = 4$ , not how to add two and two to obtain four. I am by no means denying that the addition of two and two to obtain four is significant in other senses, but what I am here chiefly concerned with is the notion that such statements as  $2 + 2 = 4$  have meaning in the way that the first sentence of this paragraph has a meaning.<sup>36</sup>

It would be foolish to claim that the meanings that can be ascribed to mathematical speech and writing are no different from meanings in ordinary discourse. Among the obvious contrasts is that between the "succinctness, generality, and precision" of mathematical language and the "compensating mix of concreteness, ambiguity, and redundancy"<sup>37</sup> of ordinary language. By indicating some of the similarities and some of the differences I intend to make a case that can easily be disagreed with but that cannot be dismissed as being about nothing at all. At the simplest level, the fact that message meanings in ordinary discourse are a person's before they are a document's is a similarity between mathematical and ordinary meaning. That this is the case has some of the same evidence as the point about ordinary meaning: one frequently does not set down on paper exactly what one meant to convey. This is a case where the writing imperfectly represents or represents not at all a meaning that could have been set down precisely but was not. When the correct or improved text is set down, one says "that is what I meant to say". To deny this categorically is to deny that anyone has had anything to say in a mathematical mode – ever. This is contradicted by even an arithmetic lesson. A point that follows from that one is that the meaning of a text is an abstraction, an interpretation, and a reader's interpretation may well differ from the author's intent. In spite of the fact that mathematical discourse is highly institutionalized – more so than ordinary language – differences of time and place still need to be taken into account in the interpretation of mathematical documents. That we take intersubjectivity for granted in our extremely varied ordinary lives makes it that much more reasonable to take it for granted in our similarly structured (at least fundamentally) mathematical lives. This basis for meaning is firmer in mathematics than elsewhere. Thirdly, persistence in considering expressions meaningless, in a language natural or mathematical, is evidence that ones does not understand the language. Fourthly, mathematics can be translated from a form imbedded in one natural language to a form imbedded in another while preserving the meaning. This is what is usually said to be preserved in a translation.<sup>38</sup>

## Objectivism and experientialism

There is in mathematics an attitude to meaning that corresponds to the objectivism that is Johnson's ordinary-language target in *The body in the mind*, namely a version of the Platonic view that there is a correspondence having nothing to do with human minds between mathematical symbols on paper and mathematical realities themselves also independent of human minds. This objectivist view attributes meaning of a sort to mathematical discourse, but of a sort that is so greatly at variance with meaning of ordinary discourse that I should hesitate to recognize it as what I mean by meaning. I cannot imagine what such a view would do with error in general or with the following meaning problem. Consider a case in which a theorem requires the hypothesis of uniform continuity but is stated with continuity. Consider two subcases, one implicitly using uniform continuity and arguing correctly, the other using only continuity and having a faulty argument. In the first case the author has meant uniform continuity, as is indicated by the use of that hypothesis in the proof. In the second subcase the author has meant continuity, perhaps not even having a notion of uniform continuity. One cannot tell from the statement of the theorem what the authors meant; that can be discovered only from their proofs, if at all. That is because the proofs may give contextual evidence for what the authors meant by the word "continuity" appearing in their theorem statements.

Meaning in ordinary language I maintained was primarily a matter of meaning in person-to-person communication and only secondarily a matter of the institutionalized meaning of texts. In mathematics, the focus is sufficiently narrow that only descriptive meanings matter, and it is almost correct to say that only the institutionalized meanings of utterances, which might as well be texts, matter to hearers and readers. For this reason, I shall ignore meanings other than the descriptive and shall not be particularly concerned with personal meaning except as the origin or interpretation of a text. The dependence of mathematics on the institutionalized meanings that come with writing is apparent from our knowledge that writing predates the creation of mathematics in the cultures that created mathematics for themselves. Put another way, I am aware of no oral culture that has produced anything mathematical beyond counting. We cannot, however, dispense with personal meanings, because they are necessary both before the writing and after the reading of any text. One can hardly expect to find evidence of the existence of personal meanings in the texts themselves; one must look at the personal remarks of mathematicians. Mathematicians of a strongly visual bent have often said that they think using pictures, and have been told by their brethren of the algebraic inclination that, while *they* may do that, not everyone does. It is therefore useful to point out that Rózsa Péter, one of the founders of recursive function theory, a subject where pictures would hardly have been useful even if she had been inclined to use them, said that

the terms, about which the axioms say something, signify nothing concrete, but rather arbitrary things that possess the basic properties stated in the axioms. However, everyone who deals with them secretly thinks about concrete things. I have already mentioned the different models that the same axioms can satisfy.

This quotation from a speech<sup>39</sup> indicates that Péter thought in terms of a model of the axiom system she worked with, that is, that the terms meant, for her, the objects of a model, which were her personal meanings of those terms at that time.

The figurative basis of language persists in mathematics. It allows us to account, as in ordinary language, for polysemy and for the development of the different senses of words. Some examples are equation, function, number, vector, limit, and space. Equations, beginning as statements in words that numbers, distances, areas, angles or whatever were equal, have been extended in algebra to statements of equality between quantities involving unknowns, between functions for specific values of argument, between functions regardless of arguments (identities), between coordinates in analytic geometry, between derivatives (differential equations), between integrals, and between functions where it is a function that is unknown. The range of what might count as an equation has expanded a great deal since the invention of the equal sign. Specific functions were in use for ages before the notion of function came on the scene. The term is linked to that of equation because of the fundamental importance of the expression  $y = f(x)$ . The word function now means variously a set of ordered pairs, its geometrical realization as the graph of the function, and the formula of the function. When one considers what a function looks like, one is certainly not thinking of the ordered pairs, and probably not of the typography of a formula; usually it is the graph of the function that gives it – by a very useful visual metaphor – an appearance. Function also is extended to refer to a dependent variable, to multiple-valued functions, functions of several variables, inverse functions, functionals, and specific kinds of object that are more fundamental (and certainly older) than function, such as sequences, which are a far cry from what the eighteenth century would have recognized as a function. Numbers, ordinal and cardinal, finite and infinite, are clearly held together by metaphorical extension, the extension to the infinite being explicitly an extension. Then there is the historical widening by metaphorical extension of the concept of number from the natural numbers to the inclusion of zero, the inclusion of negative numbers, to rational numbers (in a context of rational numbers, the symbols  $6/2$  and  $3$  have the same meaning, in contrast to a baked-goods context where  $6/2$  and  $3$  do not have the same meaning, as anyone coming home with six half pies having ordered three would tell you), to real numbers, to complex numbers and to quaternions or the alternate extension from the integers to the Gaussian integers. As a final example, I mention vectors, which are variously directed line segments in the plane, position vectors tied to the origin, the plane coordinates of the end points of the tied vectors, vector fields

still in the plane, all of these in Euclidean space, the elements of tangent spaces of manifolds, the elements of any abstract vector space (only vectors metaphorically), then linear functions on any of those. Likewise connected metaphorically are all the physical quantities that are modelled by or within vector spaces.

There is also in mathematics an analogue of the seizing of prosaic words by poets for the refinement of their meaning and then their return to enrich future prose.<sup>40</sup> The source for the major notions of mathematics is ordinary language itself, including applied mathematics. Mathematics is taught and discussed and proofs written out in largely ordinary rather than formal language. And when mathematics is applied to the world, the mathematics is embedded in ordinary discourse. From ordinary language and experience – some ordinary, some mathematical – words are rendered technical when necessary, refined in meaning (where refinement is more narrowing than expansion), and frequently returned to ordinary language, often through science. A large part of the process of the refinement of the meaning of mathematical concepts is the deduction of their interesting consequences,<sup>41</sup> the typical activity of pure mathematics. The refinement need not be a narrowing; the popular notion of number is much broader now than it was a few centuries ago when zero was a new idea and negative numbers unheard of. Quantification and set membership are notions of ordinary language that have been formalized in the last hundred years. The curvature of space-time, indeed space-time itself, is an even more recent return to common use of a notion taken from common use. Chaos is still more up-to-date and just as clearly an example of something that started in circumstances easily describable in words, was explored as a topic of pure mathematics (topological dynamics became very pure indeed), and has now been returned to a variety of ordinary circumstances such as the dripping of a tap.<sup>42</sup> Poetry that is admired is admired because it means something, not just because it sounds poetic. Likewise, theorems in mathematics are usually admired because of their meaning, rarely only because of their method of proof; this is typically why they are considered interesting, important, or deep.

While the figurative aspect of language enriches and develops mathematical language, and while analogy is the typical way in mathematics as elsewhere for us to understand things, mathematical explanation, which frequently involves argument, does not overtly use analogy in its finished form. In order for argument to be dependable, a far greater precision is required of mathematical discourse. If one thing is to be understood as being like something else, then it is necessary to say more, to determine what they have in common and to specify it sufficiently exactly that it does not matter which of the two is being talked about. If something is to be considered to be of a certain kind, then it is necessary either to assume or to prove that it is of that kind, where that kind is not a vague personal assimilation class but an institutionalized set. Much of the difference between ordinary meaning and technical mathematical meaning results from the fact that mathematics works not

with the sloppy institutionalizations of persons' assimilation classes of other persons' assimilation classes but with sets sufficiently carefully defined in writing that there is a consensus on whether any given thing is a member of any given set. This is in startling contrast to ordinary language. The notion of heap, which causes so much trouble in ordinary language, has to be made exact; if a set is to be such that the removal of one element leaves it as large as before, it must be infinite. Each person works of course with personal assimilation classes, but each of us has a lot of help in conforming our assimilation classes to the official specifications. When the official specifications are ambiguous, as convergent sequences of functions were ambiguous – some persons meaning uniformly convergent sequences and others allowing convergent sequences not uniformly convergent – error results, is detected, and the ambiguity is removed. Sometimes the specifications are even vaguer than simple ambiguity; for instance the various notions of function that were straightened out in the nineteenth century. Various persons were working for some time with each of the variety of function notions; naturally this led to a most unmathematical confusion. The conformity of our personal assimilation classes to the standard specifications is supposed to make the interpretations of mathematical texts wholly unambiguous. Most of the time it works.

We have, of course, the freedom of Humpty Dumpty to set up our own meanings for terms – without even paying them extra. But they must mean just what we choose them "to mean – neither more nor less", and we must specify in advance just what we do choose them to mean. If our meaning turns out to be attractive to others, then just like a poet, our meaning (not usually our totally new word) has caught on. It is perhaps not an accident that Alice's Humpty Dumpty was the creation of a mathematician.

### Definition

If meanings are to be found for mathematical terms, then surely the meanings are to be found in the official specifications agreed upon by the consensus of mathematicians on the basis of their private notions and the exigencies of precision to avoid contradiction. It is a fascinating feature of mathematical definitions that they define differently from the way that terms are defined in dictionaries (but see the addendum). In particular, the Aristotelian rigor means that not all terms are given definitions. A few fundamental terms in any theory will not be defined in order to avoid circularity of definition, something that is not shunned outside of mathematics. The meanings of the other, less fundamental, terms are given by definitions. The question of their being meaningless can arise from the lack of definition of the fundamental terms in terms of which they are defined. To this I have two replies.



One is that the fundamental terms of a mathematical theory have meanings to the mathematicians that know that theory; they are simply not written down as a part of the theory because to do so would be illogical. The idea that because the term point is not defined in Euclidean geometry (now) geometers no longer know what they mean by point is absurd. The logical demand that points not be defined is perfectly reasonable, but a logical demand does not eliminate knowledge. The zero-dimensional intersection of two non-parallel lines conveys the meaning of point, but to use it as an official definition would be to make the terms dimension, intersection, non-parallel, and lines all more fundamental than point, a quite unacceptable result. But it does express the meaning of a point in Euclidean geometry. Meaning is not a part of the formalism, but then why should it be? How could it be? The meaning of terms of ordinary language does not reside in those terms either.

My second reply to the suggestion that the lack of rigorous definition of the fundamental terms of a mathematical theory renders them and in consequence all its terms meaningless is that the meaning of the fundamental terms is bounded by the set of axioms of the theory taken as a whole. The axioms as a whole determine what sorts of things can satisfy them. Those sorts of thing are possible meanings for the terms; which meanings are actually used are matters of extra-formal tradition (or even personal choice). As von Glasersfeld points out,<sup>43</sup> learners (children in particular) get too little help in making these choices. The axioms of projective geometry can easily be chosen for three dimensions so that points and planes are dual concepts, so that the terms point and plane are used in exactly the same way. This means that it is possible for a person to use the term point to mean plane and the term plane to mean point throughout a theorem and its proof. Everyone else's interpretation of the theorem would be the standard one and different from that of the eccentric author, but the terms would have meanings for the eccentric and would have meanings for the readers. If the terms did not have meanings, the meanings could not differ, as they could be discovered to do by a conversation or by an oral presentation (with diagrams) of the theorem by the author. An axiom system sometimes, as this hard case shows, does not specify the meanings of its fundamental terms, even informally, but usually even the fundamental terms have meanings that are more tightly constrained by the axioms than the meanings of ordinary language are constrained by dictionary definitions. In both cases meaning is based on the mental furniture of practitioners, not on what is written down anywhere.

To the extent that the present century has created a problem of meaning in mathematics, it is demanding more of meaning in mathematics than is delivered by meaning elsewhere. Rather than mathematics being meaningless, it is just not more meaningful than ordinary language. Any demand that it should be needs to be justified. There is an irony in the fact that the study of mathematical foundations beginning with Frege has powerfully reinforced the objectivist semantics in which the demand that categories correspond, independently of us, with sets (of the rigid-

boundary mathematical kind) of things in the world. The use of computers seems always to make this absurd requirement. Given the anti-Platonic consensus that the independent existence of mathematical objects is not the case for mathematical categories, mathematics has contributed to the notion of its own meaninglessness.<sup>44</sup>

The picture should not be painted all black. Attempts have been made to give mathematics meaning in somewhat literary terms. Plato's myth of the cave, which is often interpreted as suggesting that mathematical objects can be seen, metaphorically speaking, so that mathematical statements are meaningful in a quasi-empirical sense, is used to give objectivist meaning to mathematics. A much more recent and sophisticated way of giving mathematical statements meaning is Philip Kitcher's invention<sup>45</sup> of the ideal agent, resident in the mathematical realm, who performs the operations that mathematical language so frequently refers to. Since allegory is an ancient manner of understanding what is abstract by making persons act out the rôles of abstractions,<sup>46</sup> Kitcher's ideal agent is an allegorical figure allowing an allegorical interpretation of mathematics. This is probably helpful for some persons, but it is not the only possibility.

### **Meanings, truth, and learning**

One of the reasons that mathematics has been condemned as meaningless is a second result of logical structure. Not only do we not define all our terms (thereby avoiding the circularity employed elsewhere), but also we do not ideally depend upon the meaning of our terms to ensure that our statements are true. Without bringing up the problem of truth in the abstract, everyone knows the statements that are normally taken as true in a mathematical theory, those that are assumed and those that are deduced from them. When a theory has been formalized, the correctness of conclusions does not depend on the meanings of the terms but only on the relations among them that have been assumed, the logical style that is used to make deductions, and the existence of deductions in that style leading to the conclusions. Before the formalization, which typically never does take place, the meanings of the terms are used in the all-important construction of the deductions, the insight into why the conclusions follow from the premises.<sup>47</sup> But we need to be careful that nothing in a proof depends on our personal meanings for the terms involved beyond the official meanings as circumscribed by the axioms just as we have to be careful that reasoning conducted in terms of generic particulars does not depend on their peculiarities. If this care has been sufficient, then the end result is independent of the meanings of the terms in the sense that other words could be used. We would, if we performed such an exercise of using other words, be giving new meanings to the words so used, that is official meanings as circumscribed by the axioms. We would also give them new personal meanings for the duration of the exercise, namely the

personal meanings that we formerly ascribed to the formerly used terms. The whole would still be logically valid. This is what Hilbert was illustrating with his famous remarks about talking about "tables, chairs, beer mugs" instead of "points, straight lines, planes".<sup>48</sup> In an obvious way, Hilbert's discussion assigns new meanings to the terms "table" etc.; it does not illustrate or even suggest that the terms have been emptied of meaning. If they were, the remark of F.P. Ramsey, that Hilbertism could give no account of "it is two miles to the station", would not be "silly", as Curry<sup>49</sup> calls it, but correct. As Tragesser has pointed out, Hilbert was not bound by any means to the view that there was no meaning in mathematics; he fully recognized that there was a polarity here.

On the one hand, the tendency toward *abstraction* seeks to crystalize the *logical* relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly matter. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations.<sup>50</sup>

As with ordinary language, the meanings of our fundamental mathematical words are learned from their use, and so from the things that they refer to. The number words, simple geometrical terms, names for things like quadratic equations, are learned by acquaintance with numbers of things, planar and spatial examples of geometrical objects, and work with solving equations. This is knowledge by acquaintance, by which we learn our personal meanings for most of the mathematical terms that are used in everyday life. Even the simplest kinds of these words, the number words themselves, are not all learned this way. We very soon learn how to make up these words, and then our knowledge of the larger numbers like sixteen thousand, nine hundred, and fifty-seven comes from that knowledge rather than from acquaintance. These numbers are the "same" as those that others would make up using the same rules. They gradually shift from being adjectives that might apply to hypothetical assemblages to being nouns representing mathematical objects, from being numbers of things to being just numbers. And of course we learn eventually to make up numbers so large that there are probably no assemblages in the universe which they would number. But this does not prevent "16957" from having a meaning to us.

It puzzles me how anyone can take seriously the notion that "16957" is meaningless, but some persons do say this, and so the matter is worth pursuing a short way. In mathematics, as in ordinary language, we learn the meanings of a number of words by ostensive definition, so that our whole vocabulary is ultimately ostensively defined, but we read and write sentences that do not correspond to our actual experiences. In mathematics, beginning with a few simple notions ostensively defined, like cardinality and the position of an element in an ordered set,<sup>51</sup> we proceed from counting things with adjectival numbers, through dealing with noun numbers<sup>52</sup> in

combinations in ways that are explicitly set down, to things that are not ostensibly defined but are nevertheless meaningful to us.<sup>33</sup> The rules of number-name formation allow us to make up numbers larger than we really need for counting. Noun numbers and algebra allow us to define simple functions. Once we know about functions that we can have (exhaustive) tables of or formulas for, we can understand as meaningful functions that have infinite sets as domains. Once we know about functions that we can graph finite portions of we can understand as meaningful functions that have graphs of infinite extent. Once we know about such functions, we can understand functions that are too unpleasant to be able to be represented adequately with pencil and paper and also understand conditions that allow graphs to be of the kind we formerly regarded as definitive. We can also go on to spaces of functions and operators on functions and spaces of operators on functions, and so on recursively. In this process, our increasing knowledge may recapitulate a historical process. Meanings for us grow to some extent as the meanings of the terms have grown historically.<sup>34</sup> At some point in this process we are able to stop advancing, saying, for instance, that transfinite arithmetic is not mathematics. A more correct statement would be that it goes beyond what the speaker finds meaningful. That is after all a personal decision and has to be respected while it lasts. It is, of course, subject to persuasion. Until one can attribute some meaning to the symbols that one is manipulating, one is, as a Hilbertist would claim, manipulating meaningless symbols. (On the other hand, a little manipulation can help us to understand how symbols do the jobs they are meant to do, and so understand what they mean.) Little advance in mathematics is going to be made by anyone manipulating symbols that have remained meaningless! We may think that Euler's formal and even divergent series are meaningless, but we can be fairly sure that they meant something to him. And consider how the meaninglessness of  $\sqrt{-1}$  hampered, but did not prevent, its use for centuries.

It is notorious that computers pay no attention to meaning in their algorithmic calculations. Searle's Chinese-room argument<sup>35</sup> is an example. Meaning can be evacuated from mathematics by computer-like behavior. A "vicious circle of teaching practice" has been noticed<sup>36</sup> in which schoolchildren force their teachers to "an excessive algorithmization of mathematical knowledge", causing a "disappearance' of theoretical meaning". It is obviously easier (though of course less interesting) for students to be programmed like computers than to be required to understand the meaning of what they are doing; I have noticed a tendency to expect such programming (and to be disappointed) on the part of non-mathematics undergraduates where I teach. Perhaps my students are behaviorists. As Steinbring<sup>37</sup> writes, "The development of mathematical knowledge as the extension, differentiation, and growth, of the varieties of representational structures and domains of objects means that — comfortably or not — meanings together with procedures always play a fundamental role." In the midst of a well-understood algorithm like that for the multiplication of many-digit decimal numbers with pencil

and paper, one may not attend to the meaning of what one is doing any more than a computer – until it is complete, when one may use meaning to check that the answer is reasonable.<sup>58</sup> Obviously any mathematical process that can be carried out by a computer can be dealt with in a meaningless way; but computers deal meaninglessly with text in ordinary language too (this essay is being composed with a word-processing program). As with ordinary language, if there is going to be meaning in mathematics it must be attributed by the persons involved with that mathematics.

## Reference

A second reason why mathematics may have been thought to use empty terms is the widespread acknowledgment, among those that are at all reflective, that a simple-minded Platonism that makes all mathematical terms refer to things that have locations in some highly sophisticated space is not a suitable foundation for a philosophy of mathematics. Such reference is useful in the construction of a theory of meaning, specifically that of Frege.<sup>59</sup> What we face is that the lack of this kind of reference has made some persons think that the terms involved therefore lack meaning. It is this reason that is sometimes cited as driving mathematicians to assert the radical formalism that Curry called Hilbertism. That this is backwards, basing meaning on reference instead of reference on meaning, indicates that there is a confusion here that can be cleared up. The famous phrase, "the golden mountain", fails to refer to anything literally because the meanings of golden and mountain require a conjunction of size and material that happens not to occur. The phrase can refer metaphorically to any large amount of gold or money; again, the metaphorical reference – or the reference on account of metaphorical meaning – depends on the meanings of the words concerned.

There is a second reason for reference to be of interest in considering the meaning of mathematics. Mathematics, when it is applied, is applied in virtue of meaning. In science, there is a lot of thought that can be (and has been<sup>60</sup>) described as figurative. It is quite typical that there are no interesting relations between mathematical and scientific analogues, but their internal mathematical relations can be regarded as similar to the internal physical relations; Gentner and Gentner call these relations structural relations, which are entirely distinct from the attributes of the relata. They did an interesting study of models of electricity, comparing two different ways that persons think in terms of water flow and moving crowds.<sup>61</sup> When mathematics is applied, it is usually because of similarly structural relations between the mathematical objects that are the model and the relations among the physical relata, not any similarity between mathematical objects and whatever physical corresponds to the mathematical objects. There is a tension to be felt in figurative

language; Nelson Goodman discusses it delightfully ("a metaphor is an affair between a predicate with a past and an object that yields while protesting"<sup>62</sup>). Throughout science, mathematical meaning is applied (that is, made to refer) without any of the tension that seems to be inherent in figurative language. It seems to me that the reason for the lack of tension when mathematical terms are used of physical phenomena is that, while the mathematical terms have meaning, they do not have referents attached to them as strongly as do crowds of persons or flowing water. When the physical is compared to the physical, one is constantly having to overlook the disparity between the physical systems compared. But when mathematical language is made to refer to the physical so that the physical is compared to something mathematical, one can very easily overlook the mathematical system being compared and simply use the mathematical language as referring directly to the physical. It is this fact that has eased the transition from geometry as the science of physical space to geometry as a pure mathematical subject that can be used to some degree of approximation to describe physical space. Most of the time most persons speak in geometrical terms of physical space with little if any consciousness of transfer or reference from the mathematical realm (this is a figure of speech not latent Platonism) to the physical. (Errors are often revealing. In discussing this very point, Johnson writes, "there is an identity"<sup>63</sup> where he means there is a correspondence.) Likewise persons speak in numerical terms about every sort of quantitative matter without any consciousness of transfer of reference. In both of these cases, speaking of non-mathematicians, it may be that there is no transfer, that the only meanings that some persons have of geometrical and arithmetic terms are spatial and quantitative. If this is the case, there is probably a transition that is difficult for most persons from thinking only spatially and quantitatively when doing mathematics to thinking geometrically and arithmetically — not to mention algebraically, topologically, logically, and so on. It would then be counterproductive to make constant references to the concrete in teaching about abstractions, as Pimm claims has become "an increasingly common dogma" at least in English schools.<sup>64</sup>

Application of mathematics, thought of in this way, is not the mystery alluded to by Eugene Wigner<sup>65</sup> and many others, but the natural result of mathematics being "formal systems which have arisen from real human activities",<sup>66</sup> as Saunders MacLane puts it, or "the study of the structures that we use to understand and reason about our experience",<sup>67</sup> as Lakoff puts it so broadly as to include grammar. If, as I think, mathematical meaning is rooted, as Johnson shows ordinary language is, in our physical and then mental experiences, its applicability to those experiences (though not any sort of correctness) is tautologous.<sup>68</sup> It is quite clear to me that a good deal of our more sophisticated experience is constructed in terms of mathematical application so that the two constructions, of mathematics and experience, go on together in interdependence.

It seems that there are degrees of reference. "The cat sat on the mat" refers less particularly than "Jennyandots sat in the warm and sunny spot", where no particular cat is in mind in the first sentence and the second refers to a specific cat. The mathematics of  $2 + 2 = 4$  is said to be applied when it is said that  $\$2 + \$2 = \$4$ , but it is still more applied (refers more particularly) if one is actually making change. As this example illustrates, a pure mathematical notion like two can be removed farther from reference than can a word like cat. But their meanings dictate equally what might be called their possible literal referents. "Two cats" is as inappropriate for one cat as for two dogs.

I conclude these remarks on reference with one of the differences, as it seems to me, between ordinary language and mathematical language. I have emphasized that the interpretation of ordinary language is fundamentally guesswork, however well educated and correct. The independence of meaning that allows formalism in mathematics has an effect here. When reading a mathematical text (or hearing a talk; it matters very little), one is told explicitly, or it is assumed that one knows, the mathematical system in which the writer is writing. Given that, one is free to interpret what is written in terms that are meaningful, whether they are those that the writer had in mind or not. One does not much care because it does not much matter.<sup>69</sup> If the meaning of a theorem and the argument of its proof were in one model for the writer and the reader interprets the theorem and argument in terms of another model, nothing has been lost, because it was the theorem and the argument that the writer was trying to convey, not the particular interpretation. When the argument is easier to understand for some readers in one interpretation, for instance a geometric interpretation, then the writer may indicate this. But usually, a convincing argument is convincing regardless of personal variation in how one thinks about it (the meaning and consequent reference one attributes to it). This is because mathematics is always more interested in relations among relations than in the relations themselves and still more than in the relata. In different mathematical contexts, not to mention applications, a theorem can mean a variety of different things, and one uses one's meaning for it to understand the theorem and its proof. This is in marked contrast to the situation in ordinary language where the apparatus of speech is there to convey the meaning; one understands a sentence in order to grasp its meaning and reference.

## Conclusion

In conclusion, I want to point out that in this essay I have been trying to accomplish two separate things. First, to present a sketch of a notion of meaning that is of independent interest and not as well known as I think it should be without elaborating a theory of meaning. I take this view of course because I came

independently to some of it before finding it expounded by Pepper, Lakoff, and Johnson, to whom reference should be made. Second, to present, in terms of the notion of meaning sketched, a case against the harmful idea that mathematics is meaningless. It is, I think, not necessary that one should swallow whole the discussion of the first part in order to accept the second. Before one can cogently argue that mathematics is meaningful, one has to have some notion of meaning in terms of which to say anything. What I have tried to do is to show that, in terms of my notion of meaning, mathematics can be meaningful. If one wishes to argue that mathematics cannot be meaningful, which has tended not to be argued but merely assumed, and which is a stronger statement than that mathematics can fail to be meaningful, then for the weak attack one needs either to show how mathematics is meaningless on my terms or if one rejects my terms then to present another notion of meaning in terms of which ordinary language is meaningful and mathematics is not. For the strong attack one needs to show that one cannot have a notion of meaning in terms of which ordinary language and mathematics are both meaningful. It is not good enough just to assume that mathematics is not meaningful; it is neither a fruitful nor an attractive axiom.

### **Part 3. Addendum**

A third part of this essay plays no rôle in the first two parts but takes up two lacks that are mentioned there without apology. No connection is made between the meanings of words and the meaning of life. And no specific notion of meanings of words is given. The reasons for these two lacks are opposite; the latter is because I think that the main argument must be independent of the specifics and even the existence of a theory of meaning, not because I do not have one, while the former is because I only recently saw how one might connect the meanings of words and the meaning of life. In this part I shall set out briefly an idea of word meaning and conclude with an indication of how that idea can connect word meaning and life meaning.

### **Words and mathematics**

My idea of word meaning is set out here to answer the charge that the merely implicitly defined meaning of a term in mathematics is properly contrasted to the ostensive definition that predominates in ordinary language. This is an apparent weakness in my claim of parallelism between meanings in mathematics and in ordinary language. I am particularly eager to combat this charge because I think that it is an important point about ordinary-language meaning as well as about mathematical meaning.



Let us consider for a moment the mathematical flank of the charge. The meaning of point in Euclidean geometry is intersubjectively defined – in so far as it is defined – by the specification that a point bears certain relations to the other entities in the theory. I do not wish to avoid this fact about the official definitions in mathematics. One's personal meaning is much more specific and is based on the geometrical tradition or on personal choice, but one's personal meaning is not what matters. What is needed is to suggest how the mode of official mathematical meaning can be squared with ordinary meaning, where the personal meanings are fundamental. I do this by claiming that personal meanings are also determined by networks of relations to other classes of entity.

One should recall that what one means by a word, according to what I said in the Reference section of Part 1, has to do with a personal disposition "to place things we have or have not before met into classes of things we already know about". In other words, one uses a word to describe or refer to something if it is, for the present purpose, to be regarded as of the kind that the word refers to; that kind of thing (or occasion or relation or what have you) is what one means by that word. Likewise, that kind of thing is what one understands by another's use of the word. One uses that word rather than another because one is related to the thing (or occasion etc.) in a way that one feels it is unnecessary to distinguish from the way that one is related to the other things in the assimilation class represented by the word. Necessary distinctions are displayed in modifying words, phrases, or clauses. Because the assimilation classes with which each one of us works are the personal creations of each of us, this relation between speaker and referent is not subject to outside comment; it is what it is. If I want to say that something is on a par with other books, then I call it a book; if I consider that a situation would be annoying except that it is funny, then it is clearly not quite on a par with other annoying situations, and this can be signalled by the addition of the modifying clause. While word choice is dependent on the relation I have just mentioned, there is a way in which it depends on relations more like the relations that define mathematical terms.

The way in which I would normally speak of a thing's being a book is not to claim that it has books' relation to me but rather to objectify its bookishness and say that it has a relation of similarity to other books (to the point of not being distinguished from them). Something is red if it is similar in color to red things. Something is winning if it is similar, in relation to race or to competitors, to other winnings of races or competitions. It is of some importance that these similarities allow for the metaphorical uses of words as well as their literal uses; the relation is similarity not identity. At the metaphorical extreme, I may wish to draw attention to only a single similarity between the thing to which I refer and the category in which I verbally place it. Something is a book or red or winning because it looks (or seems to some other sense) like other books, other red things, or other victories. I do not mean by "similar" "partially identical", which Austin said was its meaning in

the classical objectivist view of categories.<sup>70</sup> One looks in vain for some set of essential features of books or redness or victory (the objectivist attempt shown to be hopelessly inadequate by Lakoff and by Johnson, gathering evidence from other workers). The most that we can find is a family resemblance, even among the things to which all of us would apply a word. Family resemblance is a relationship, is a relationship, nothing more or less. And as Lakoff points out, other ways in which meaning works, like metonymy, depend on relations. So far I have written chiefly about categorization, but ordinary-language meaning's basis in relation is even more clear in higher-level meanings. More complicated schemata that we mean when we describe situations in words are, as Johnson points out and explains in some detail, more subtle than just propositions that we match to what we wish to describe. There is in each image schema a complex of relations in some realm of experience, and when we use that image schema we are drawing upon analogy between what we are describing and that schema (again not partial identity). When explaining how schemata differ from images on the one hand and propositions on the other, Johnson uses Kant's example of the mathematical notion triangle as something that is not appropriately captured as an image but as a schema.<sup>71</sup> The same is still clearer in the other examples he gives like the (human) body as a machine. Generally, schemata have parts in relation. The application of a schema to a situation depends on some analogy between the relations in the schema and the relations in the situation; all that is required of the parts is that they be matched up with the entities in the situation. No analogy of themselves is required. This is the same as when mathematics is applied. It is not usual that the mathematical entities are in any way analogous to the entities in the area of application. What matters is that the situational relations are appropriately modelled by the mathematical relations. And this corresponds of course to the fact that in the mathematics it is only the relations among the mathematical entities that matter. The importance of the application of image schemata to Johnson's understanding of "meaning, understanding, and rationality" (to quote from the title of his fifth chapter) cannot be exaggerated.

That the above fits well with the application of mathematics is not surprising, but it is mentioned to support the notion that it fits well with ordinary language. If one gives up the notion that there are essential features of something that (objectively) make it a book, and essential features of all books that make them books, then what you are left with is a relational notion of what makes something a book; if bookishness is not how the book is in itself, then bookishness for me must be how the putative book is situated in the world as I see it, and objective bookishness must be how the putative book is situated in the world as persons in general see it. Objective bookishness (interpersonally, not classically) is a relation among different persons' world views based upon relations within those world views. Typically the latter relations are never voiced, only the resulting judgments of bookishness, which allow the analogy among various persons' world views as they pertain to the book to be discovered. If this looks a bit like the relations rather than essential features that

make something a point in geometry, then this defense has been successful. I am not trying to argue for identity, only that an analogy is not far fetched.

The most radical statement of this thesis would be that there are not properties, only relations. I do not make this statement because there is, I can see, a use for speaking of properties, and because I am not trying to do metaphysics. I should say only that anything that can be said by speaking of properties can be said, perhaps less succinctly, in terms of relations. That this is a natural view for a mathematician was observed by Scott Buchanan, who wrote,

The mathematician sees and deals with relations; the poet sees and deals with qualities. Functions and adjectives respectively are the symbols through which they see and with which they operate. Mathematics is analytic, seeing wholes as systems of relations; poetry is synthetic, seeing wholes as simple qualities. The qualities that the poet sees are due to relations, says the mathematician. They need purgation. The relations that the mathematician sees are concrete and factual, says the poet. They need appreciation and love.<sup>72</sup>

I shall conclude this defense by pressing the point. For mathematics, things are defined in terms of their relations, but this is the formality. The meanings for persons are typically more than the formalities. In ordinary language, personal meanings are primary and more than shared meanings, containing many nuances of personal experience. The common meanings are the family resemblances among these personal meanings, again less than the personal meanings and corresponding to the more formal common meanings in mathematics. The family resemblances are relations relating words and schemata of one's personal world and the words and schemata of others' personal words. Point is defined implicitly by the rôle that points play in an axiom system. Red is defined implicitly by the role that "red" plays in conversations, which reflects the relations that red things have in the worlds of those conversing. Which relations? Color relations.

### **More general meanings**

Now that my idea of word meaning is available for comparison, I can give some substance to Johnson's notion, mentioned at the beginning, that the meaning of lives need not be wholly different from the meaning of words. The connection requires an interpretation of the phrase "meaning of life" that is new to me but at first sight has made some sense. Erich Jantsch suggests<sup>73</sup> that to grasp the meaning of one's life is to grasp the role that one plays in the universe. That knowing the meaning of a word is to know the relations to the rest of the universe that something would have before the word would be applied to it is consonant with its notion. This is not the

place to explore it. As usual, fictional lives can be as meaningful as real ones, and invented incidents make up as much meaning in lives as the things a person really did and suffered. The Becketts of the medieval Church, of T.S. Eliot, and of Jean Anouilh have meanings that the life of the archbishop may not have had.

I want to acknowledge gratefully the benefit of many hours of conversation on the topic of Part 1 with Jim Romeyn; this is not to say that he agrees with it all.

## NOTES

1. "Inquiry into meaning and truth," *Phil. Math.* II, 5 (1990), 73-87.
2. Harper and Row, 1964.
3. University of Chicago Press, 1987.
4. University of Chicago Press, 1987.
5. Oxford University Press, 1936.
6. *A midsummer-night's dream*, Act V, Scene 1.
7. I owe this example to George Steiner, *Real presences* (University of Chicago Press, 1989), p. 195.
8. W.H. Leatherdale, *The role of analogy, model and metaphor in science* (North-Holland, 1974).
9. Janet Martin Soskice, *Metaphor and religious language* (Oxford: Clarendon Press, 1985).
10. University of California Press, 1942.
11. *Concept and quality: a world hypothesis* (Open Court, 1967).
12. With George Lakoff (Chicago University Press, 1980). See also George Lakoff and Mark Turner, *More than cool reason: a field guide to poetic metaphor* (Chicago University Press, 1989).

13. Theodore R. Sarbin, "The narrative as a root metaphor for psychology", in Theodore R. Sarbin (ed.), *Narrative psychology: the storied nature of human conduct* (Praeger, 1986), pp. 3-21.
14. Renford Bambrough, "Universals and family resemblances", *Proceedings of the Aristotelian Society*, LXI (1960-61), pp. 207-222. The *locus classicus* for this approach is Wittgenstein's *Philosophical investigations* (Blackwell, 1953).
15. *Introduction à l' épistémologie génétique* (Paris, 1950) and many other works, many translated into English. Epistemology in Piaget's sense has little to do with the justification of knowledge and almost everything to do with the growth of it. For Piaget, see J.H. Flavell, *The developmental psychology of Jean Piaget* (Van Nostrand Reinhold, 1963) and Richard F. Kitchener, *Piaget's theory of knowledge* (Yale University Press, 1986).
16. *Op. cit.*
17. E.g., I. Gati and A. Tversky, "Weighting common and distinctive features in perceptual and conceptual judgments", *Cognitive Psychology*, 16 (1984), 341-370, and references.
18. For the fuzzy set theory of Zadeh in this context, see Lakoff, *op. cit.*, pp. 196f. Körner years ago had shown that ordinary set theory would not do this job in *Conceptual thinking: a logical inquiry* (Dover, 1959).
19. This has been discussed by W.V.O. Quine, *Word and object*, M.I.T. Press, 1960. Earlier was L. Pirandello in *Six characters in search of an author*, where a character says

But don't you see that the whole trouble lies here. In words, words. Each one of us has within him a whole world of things, each man of us his own special world. And how can we ever come to an understanding if I put in the words I utter the sense and value of things as I see them; while you who listen to me must inevitably translate them according to the conception of things each one of you has within himself. We think we understand each other, but we never really do.

Quoted in Scott Johnson, *Six characters in search of an author: a constructivist view*, *Family Process*, 29 (1990), 297-308 from E. Bentley (ed.), *Naked masks* (E.P. Dutton, 1952). Still earlier was Pascal (*Pensées*, #392 in the Modern Library translation by W.F. Trotter) on the meaning of terms:

We assume that all conceive of them in the same way; but we assume it quite gratuitously, for we have no proof of it. I see, in truth, that the same words are applied on the same occasions, and that every time two men see a body change its place, they both express their view of this same fact by the same word, both saying that it has moved; and from this conformity of application we derive a strong conviction of a conformity of ideas. But this is not absolutely or finally convincing...

Of the five uses of "the same" in this quotation, only the fourth expresses identity rather than assimilation, and even that can be questioned. A sixth use has been avoided by using the word "conformity". (This is fragment 109 in editions of L. Lafuma published by Luxembourg [1951] and Seuil [1963].)

20. Research by Eleanor Rosch is summarized with a dozen references in Lakoff, *op. cit.*
21. Pepper's term for this situation in *Concept and quality*, Sect. 13.4.
22. E.g. James J. Kaput, "Towards a theory of symbol use in mathematics" in *Problems of representation in the teaching and learning of mathematics*, Claude Janvier, ed., (Lawrence Erlbaum, 1987), p. 162. Throughout his paper, Kaput takes as given the point of this paper, that mathematical language is meaningful. He claims to be doing "philosophical engineering" (p. 171).
23. I say fits rather than matches in agreement with von Glasersfeld because I agree with him that match is inappropriate, among other reasons because I see nothing to match. For example, Ernst von Glasersfeld, "Learning as a constructive activity" in Claude Janvier (ed.), *ibid.*, pp. 3-17.
24. Steiner, *op. cit.*, p. 90, points out that in the Garden of Eden, Adam has such access. It is clearly mythical, even for Plato.
25. R. Rommetveit, "Language acquisition as increasing linguistic structuring of experience and symbolic behavior control" in J.V. Wertsch (ed.), *Culture, communication, and cognition*, Cambridge University Press, 1985, pp. 183-205, quoted in Paul Cobb, *Experiential, cognitive, and anthropological perspectives in mathematics education*, *For the learning of mathematics*, 9 (2) (1989), 32-42.
26. *Concept and quality*, p. 450.

27. Steiner, *op. cit.*, p. 82, writes

The informing matrix or context of even a rudimentary, literal proposition – and just what does *literal* mean? – moves outward from specific utterance or notation in ever-widening concentric and overlapping circles. These comprise the individual, subconsciously quickened language habits and associative field-mappings of the particular speaker or writer. They incorporate, in densities inaccessible to systematic inventory, the history of the given and of neighboring tongues. Social, regional, temporal, professional specificities are of the utmost relevance. As the ripples and shot-silk interference effects expand outward, they become of incommensurable inclusiveness and complexity.

28. This point was discussed recently by Robert S. Tragesser, "Sense perceptual intuition, mathematical existence, and logical imagination", *Phil. Math.* II, 4 (1989), 154-194, especially p. 180, where he draws the discomfoting conclusion, that, in ordinary language, "sustained, assuredly valid deductive inference [is] infeasible" and its corollary that it is feasible only in mathematics (p. 182). In Johnson's terms, mathematical reasoning is the product of limiting what is said (but not necessarily thought) to what is sufficiently disconnected from embodiment to allow dependable deduction. Rigor is this limitation and is usually imperfect. Johnson thinks (*op. cit.*, p. 38) that it is done "quite thoroughly" in mathematics. Steiner (*op. cit.*, p. 95) calls "a vulgar illusion" the ascription to words of "a correspondence to 'things out there'."
29. The attentive reader will have noticed that I have not rigorously separated the discussion of reference from that of meaning. This is, I hope, not a confusion, but a reflection of a necessity like that, in a discussion of bathing, of failing rigorously to separate discussion of the baby from discussion of the bathwater. Such a separation would miss the point.
30. Bertrand Russell, *Inquiry into meaning and truth* (Penguin, 1967), p. 290.
31. Both pictures and music seem to me able to refer without attaching any meaning to the reference; both suggest significance.
32. Putnam has argued that no objective reference relation can exist in his *Reason, truth, and history* (Cambridge University Press, 1981), used extensively by Lakoff. The above argument is that, even if there were any such relation, it would not matter.
33. Keith Donnellan, "Reference and definite descriptions", *Philosophical Review*, 75 (1966), 281-304.

34. That mathematics has significance in this larger sense through symbolism has recently been argued by Sarah Voss, "Depolarizing mathematics and religion", *Phil. Math.* II 5 (1990), 129-141.
35. The important harm that this notion has done in mathematics is spelled out by Reuben Hersh. Some proposals for reviving the philosophy of mathematics, *Advances in math.*, 31(1979), 31-50; some of the worst damage it has done has been outside mathematics. The consequences are obviously catastrophic for mathematical education if one of its important purposes is to make mathematics meaningful. Precisely this is maintained by Kaput, *op. cit.*, pp. 172, 175, and I agree. Learners need to be moving in the other direction, actively seeking meaning. As Hassler Whitney wrote (*Mathematical reasoning: early grades*, Growth through involvement, curriculum outline, May, 1988, quoted Anneli Lax, Hassler Whitney, 1907-1989: Some recollections, 1979-1989, *Humanistic Mathematics Network Newsletter #4*, December, 1989, 2-7):
- ...the "mathematical" experience of most children these days, especially in the inner city, is one of trying to learn the rules of the day; they give up seeing meanings somewhere in the early grades. The great need for children is to return to their wonderful preschool learning, when they were full of vitality and curiosity, exploring their environment, observing a myriad of interconnections, and learning complex concepts and skills like communicating verbally and nonverbally, beyond what any of us adults do, and without any formal teaching.
36. The approach that Johnson takes to meaning especially relevant to mathematics is exemplified in his discussion of equality in terms of the schema of balance in Chapter 4.
37. Kaput, *op. cit.*, pp. 159-195.
38. The term "translation" is sometimes used for the transition from formula to sketch of a function. In view of the editing that takes place when the domain is infinite, this is certainly no better than paraphrase preserving at most the most salient features. E.g., Claude Janvier's chapters in Claude Janvier, (ed.), *ibid.*
39. Speech to teachers and students in Rostock, East Germany in 1963, published in *Mathematik in der Schule*, 2 (1964), 81-90, and translated as "Mathematics is beautiful," *Mathematics Intelligencer*, 12 (1990), No. 1, 58-64.



40. The relation of poetry and mathematics is one of perennial interest. Most recently, W.M. Priestley, "Mathematics and poetry: How wide the gap?" *Mathematical Intelligencer*, 12 (1990), No. 1, 14-19. Most famously, Scott M. Buchanan, *Poetry and mathematics* (University of Chicago Press, 1962), originally published in 1929.
41. There is an interesting reflection here of the memorability of poetry; mathematics that is not worked out because it is interesting but only because it is needed for some application is frequently forgettable and forgotten. Hardy's disparaging remark about ugly mathematics could as easily have been said of unmemorable mathematics; they are the same thing.
42. James Gleick, *Chaos: making a new science* (Viking Penguin, 1987).
43. *Op. cit.*
44. Lakoff, *op. cit.*, pp. 177-179 and Chapter 14.
45. *The nature of mathematical knowledge* (Oxford University Press, 1984).
46. Dorothy L. Sayers, "The writing and reading of allegory" in *Christian letters to a post-Christian world*: Eerdmans, 1969).
47. Gila Hanna, Some pedagogical aspects of proof, *Interchange*, 21 (1990), 6-13, observes that it is typical of mathematicians to accept proofs "for their semantic rather than their syntactic qualities." Another speaker at the same conference (J.L. Berggren, "Proof, pedagogy, and the practice of mathematics in medieval Islam", *Interchange*, 21 (1990), 36-48) quoted Bourbaki ("The architecture of mathematics", translated by R.A. Hall and H.G. Bergmann, in F. Le Lionnais (ed.) *Great currents of mathematical thought*, Vol. 1. (Dover)) to the same purpose:
 

A proof is not really "understood" as long as [the mathematician] has only verified the correctness of the deductions involved step by step, without trying to understand clearly the ideas which led to the construction of this chain of deductions in preference to all others.
48. Conversation in Berlin railway waiting room reported by Otto Blumenthal. The reference "D. Hilbert *Gesammelte Abhandlungen*, vol. 3, pp. 402-403" is given in Howard Stein, "*Logos, Logic and Logistiké*: Some philosophical remarks on nineteenth-century transformation of mathematics" in William Aspray and Philip Kitcher, *History and Philosophy of modern mathematics* (University of Minnesota Press, 1988), pp. 238-259.

49. Haskell B. Curry, *Outlines of a formalist philosophy of mathematics* (North-Holland, 1958), p. 67.
50. Preface signed by D. Hilbert, to D. Hilbert and S. Cohn-Vossen, *Geometry and the imagination*, trans. P. Nemenyi (Chelsea, 1952), referred to in Tragesser, *op. cit.*
51. Piaget-style research shows that children can know that physical manipulation preserves the numerical measure of the cardinality of a finite set but at the same time makes it larger or smaller (J. Bergeron and N. Herscovics, "A model to describe the construction of mathematical concepts from an epistemological perspective" in Lionel Pereira-Mendoza and Martyn Quigley, Proceedings of the 1989 annual meeting of the Canadian Mathematics Education Study Group/Groupe canadien d'étude en didactique des mathématiques, Brock University, St. Catharines, Ont., 1989, pp. 99-114). Official meaning is hard won.
52. This is a transition that can cause trouble even when it has apparently been negotiated successfully; see Kaput, *op. cit.*, pp. 163, 188.
53. As with ordinary language, it is all more useful if it conforms to standard patterns.

For example, a child picking up elementary counting might not like the number 8 for some reason and so always elide 8 from all sequences. Thus elementary counting becomes 1, 2, 3, 4, 5, 6, 7, 9, 10. This is a perfectly possible sequence, since the designations are arbitrary. But it is not the standard sequence. Thus all counting, elementary arithmetic, and elementary algebra are affected. Proofs are not understood in a host of instances, and going on in arithmetic indefinitely is blocked nearly everywhere. ...But such cases are observed in the classroom every day. And the children who count or multiply like this are considered stupid. Perhaps some are. (Ian Winchester, "Understanding mathematical proof – beyond 'Eureka!'," *Interchange*, 21 (1990) 65-71).

54. John M. Mason discusses this process, which he calls symbolizing, in "When is a symbol symbolic?" *For the learning of mathematics* 1 (1980), 8-12 and "What do symbols represent?" in Claude Janvier (ed.), *ibid.*, pp. 73-81. The recursiveness of the process is drawn attention to in Susan Pirie and Tom Kieran, A recursive theory of mathematical understanding, *For the learning of mathematics*, 9 (1989), 7-11, which is based on work of von Glasersfeld and H. Maturana.

55. John R. Searle, "Is the brain's mind a computer program?" *Scientific American* 262 (1990) No. 1, 26-31, and "Minds, brains, and programs" in *Behavioral and brain sciences*, 3 (1980), 417-458.
56. Heinz Steinbring, "Routine and meaning in the mathematics classroom", *For the learning of mathematics*, 9 (1989), 24-33. This paper is concerned with preserving, indeed enhancing, meaning in the schoolroom.
57. *Op. cit.*
58. For a horrifying real schoolroom example, see (if experience does not serve) the example of the two roots of  $\sqrt{x} = -x + 6$  on page 112 of "Pedagogical considerations concerning the problem of representation" by Bernadette Dufour-Janvier, Nadine Bednarz, and Maurice Belanger in Claude Janvier (ed.), *ibid.*, pp. 109-122.
59. Sketched in Johnson, *op. cit.*, pp. xxx, xxxi.
60. Leatherdale, *op. cit.*
61. Dedre Gentner and Donald Gentner, "Flowing water or teeming crowds: Mental models of electricity", in Dedre Gentner and Albert Stevens (eds.), *Mental models* (Lawrence Erlbaum, 1983), 99-129. It is used as an example by Johnson, *op. cit.*, Chapter 5.
62. *Languages of art* (Bobbs-Merrill, 1968), p. 69.
63. P. 115.
64. "Natural language and mathematical language", in Lionel-Pereira-Mendoza and Martyn Quigley, *ibid.*, pp. 61-69.
65. Eugene P. Wigner, The unreasonable effectiveness of mathematics in the natural sciences, *Comm. pure & app. math.* 13 (1960), 1-14. "Why it should be that the external world, in the naïve, obvious sense, should concur with the regularity-postulates, with the mathematical and rule-bound expectations of investigative rationalism, no one knows." Steiner, *op. cit.*, p. 71.
66. Saunders MacLane, "Mathematical models: A sketch for the philosophy of mathematics", *Amer. Math. Monthly*, 88 (1981), 462-472. His view is elaborated in *Mathematics: form and function* (Springer, 1986).

67. Lakoff, *op. cit.*, p. 355, where the above quotation from MacLane appears.
68. I must remark that the earliest expression of this sort of notion of mathematical applicability that I know of is in the *Anti-Düring* of F. Engels. I owe this reference to V.A. Panfilov, "Philosophical questions of mathematics in *Anti-Düring*", *Phil. Math.* II, 4 (1989), 147-153.
69. Tragesser, *op. cit.*, calls the feature of mathematical concepts that allows them to "be looked at in more than one way" their robustness. It is an important property.
70. John L. Austin, "The meaning of a word" in *Philosophical papers* (Oxford University Press, 1961), referred to in Lakoff, *op. cit.*, p. 18.
71. Johnson, *op. cit.*, p. 24.
72. Scott M. Buchanan, *Poetry and mathematics* (Chicago University Press, 1962), p. 135.
73. *The self-organizing universe* (Pergamon Press, 1980), p. 310, quoted in Matthew Fox, *The coming of the cosmic Christ* (Harper & Row, 1988), p. 253.

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