

Mathematics and games

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### **Mathematics is not a game**

As a mathematician I began to take an interest in philosophy of mathematics on account of my resentment at the incomprehensible notion I encountered that mathematics was 'a game played with meaningless symbols on paper', not a quotation to be attributed to anyone in particular, but a notion that was around before Hilbert. [1] Various elements of this notion are false and some are also offensive. Mathematical effort, especially in recent decades, and the funding of it indicate as clearly and concretely as is possible that mathematics is a serious scientific-type activity pursued by tens of thousands of persons at a professional level. While a few games may be pursued seriously by many and lucratively by a professional few, no one claims spectator sports are like mathematics. At the other end of the notion, paper is inessential, merely helpful to the memory. Communication, which is what the paper might hint at, *is* essential; our grip on the objectivity of mathematics depends on our being able to communicate our ideas effectively. Turning to the more offensive aspects of the notion, we think often of competition when we think of games, and in mathematics one has no opponent. Such competitors as there are are not opponents. Worst of all is the meaninglessness attributed to the paradigm of clear meaning; what could be clearer than  $2 + 2 = 4$ ? Is this game idea not irredeemably outrageous?

Yes, it is outrageous, but there is within it a kernel of useful insight that it is the purpose of this essay to point to because it is so easily and so often obscured by outrage at the main idea, not often advocated presumably for that reason. I know of no one that claims that mathematics is a game or bunch of games. The main advocate of the idea that doing mathematics is *like* playing a game is David Wells. [3]

It seems that no philosophical idea is entirely without a kernel of insight. Mathematics is not a collection of games, but perhaps it is somehow like games, as written mathematics is somehow like narrative. I have been persuaded of the merit of some comparison with games in two stages, during one of which I noticed a further fault with the notion itself: there are no meaningless games. Meaningless activities like tics and obsessions are not games and no one mistakes them for games. Meanings in games are internal, not having to do with reference to things outside the game as electrons, for example, in physics are supposed to refer to electrons in the world. The kings and queens of chess would not become outdated if all nations were republics.

The 'meaningless' aspect of the game notion is self-contradictory; it might be interesting to know how it got into it and why it stayed so long. Taking it as given then that games are meaningful to their players and often to spectators, how are mathematical activities like game-playing activities? The first stage of winning me over to a toleration of this comparison came in my study of the comparison with narrative. [4] One makes sense of narrative, whether fictional or factual, by a mental construction that is sometimes called the world of the story. Keeping in mind that the world of the story may be the real world at some other time or right now in some other place, one sees that this imaginative effort is a standard way of understanding things that people say; it need have nothing at all to do with an intentionally creative imagining like writing fiction. In order to understand connected speech about concrete things one imagines them. This is as obvious as it is unclear how we do it. We often say that we pretend that we are in the world of the story. This pretence is one way—and a very effective way—of indicating how we imagine what one of the persons we are hearing about can see or hear under the circumstances of the story. To put it in personal terms, if I want to have some idea what a person in certain circumstances can see, for example, I imagine myself in those circumstances and ask myself what I can see. [5] Pretending to be in those circumstances does not conflict with my certain knowledge that I am listening to the news on my radio at home. This may make it a weak sort of pretence, but it is no less useful for that. The capacity to do this is of some importance. It encourages empathy, but it also allows one to do mathematics. One can pretend what one likes and consider the consequences at any

length entirely without commitment. This is often fun, and it is a form of playing with ideas. Some element of this pretence is needed, it seems to me, in changing one's response to 'what is  $2 + 2$ ?' from ' $2 + 2$  what?' to the less concrete 'four'. [6]

This ludic aspect of mathematics is emphasized by Brian Rotman in his semiotic analysis of mathematics [7] and acknowledged by David Wells in his comparison of mathematics and games. Admitting this was the first stage of my coming to terms with games. The ludic aspect is something that undergraduates, many of whom have decided that mathematics is either a guessing game (a bad comparison of mathematics and games) or the execution of rigidly defined procedures, need to be encouraged to do when they are learning new ideas. They need to fool around with them to become familiar with them. Changing the parameters and seeing what a function looks like with that variety of parameter values is a good way to learn how the function behaves. And it is by no means only students that need to fool around with ideas in order to become familiar with them. Mathematical research involves a good deal of fooling around, which is part of why it is a pleasurable activity. This sort of play is the kind of play that Kendall Walton illustrates with the example of boys in woods not recently logged pretending that stumps are bears. [8] This is not competitive, just imaginative fooling around.

I do not think that this real and fairly widely acknowledged—at least never denied—aspect of mathematics has much to do with the canard with which I began. The canard is a reductionistic attack on mathematics, which said to be 'nothing but' something it is not: the standard reductionist tactic. In my opinion, mathematics is an objective science but a slightly strange one on account of its subtle subject matter; in some hands it is also an art. [9] Having discussed this recently at some length, [10] I do not propose to say anything about what mathematics *is* here but to continue with what mathematics is *like* because such comparisons, like that with narrative, are instructive and sometimes philosophically interesting. The serious comparison of mathematics with games is due in my experience to David Wells, who has summed up what he has been saying on the matter for twenty years in a strange document, draft zero of a book or two called *Mathematics and abstract games: An intimate connection*. [3] Wells is no reductionist and does not think that mathematics is any sort of a game, meaningless or otherwise. He confines himself to the comparison ('like a collection of abstract games'—p. 7, a section on differences—pp. 45–51), and I found this helpful in the second stage of finding insight in the comparison. But I did not find Wells's direct comparison as helpful as I hope to make my own, which builds on his with the intent of making it more comprehensible and attractive (*cf.* my opening sentence).

### **Doing mathematics is not like playing a game**

Depending on when one thinks the activities of our intellectual ancestors began to include what we are prepared to acknowledge as mathematics, one may or may not include as mathematics the thoughts lost forever of those persons with the cuneiform tablets on which they solved equations. The tablets themselves indicate procedures for solving those particular equations. Just keeping track of quantities of all sorts of things obviously extended back in time to well before anything we would recognize as mathematics and gave rise to arithmetic. Keeping track of some of the many things that one cannot count presumably gave rise to geometrical ideas. It does seem undeniable that these procedural elements are the historical if not the logical basis of mathematics, and not only in the near east but also in India and China. I do not see how mathematics could arise without reflection on such pre-existing procedures—probably written down, for it is so much easier to reflect on what is written down. This consideration of procedures and of course their raw material and results is of great importance to my comparison of mathematics and games because *my comparison is not between playing games and doing mathematics*. I am taking mathematics to be the sophisticated activity that is the subject matter of philosophy of mathematics and research in mathematics rather than actions like adding up columns of figures and more complicated actions happily transferred from humans to computers. Mathematics is what we want to keep for ourselves. When playing games we stick to the rules (or we are changing the game being played), but doing serious mathematics (not executing algorithms) we make up the rules—definitions, axioms, and some of us even logics. As Wells points out in the section of his book on differences between games and mathematics, in arithmetic we find prime numbers, which are a whole new 'game' in themselves (metaphorically speaking).

While mathematics requires reflection on pre-existing procedures, reflection on procedures does not become recognizable as mathematics until the reflection has become sufficiently communicable to be convincing. Conviction of something is a feeling, and so it can occur without communication and without verbalizing or

symbolizing. But to convince *someone else* of something we need to communicate, and that does seem to be an essential feature of mathematics, whether anything is written down or not, a fortiori whether anything is symbolic. And of course convincing argument is proof. The analogy with games that I accept is based on the possibility of convincing argument about abstract games. Anyone knowing the rules of chess can be convinced that a move has certain consequences. Such argument does not follow the rules of chess or any other rules, but it is *based* on the rules of chess in a way different from the way it is based on the rules of logic that it might obey. To discuss the analogy of this with mathematics I think it may be useful to call upon two ways of talking about mathematics, those of Philip Kitcher and of Brian Rotman.

### **Ideal Agents**

In his book, *The nature of mathematical knowledge* [11], Kitcher introduced a theoretical device he called the ideal agent. ‘We can conceive of the principles of [empirical] Arithmetic as implicit definitions of an ideal agent. An ideal agent is a being whose physical operations of segregation do satisfy the principles [that allow the deduction in physical terms of the theorems of elementary arithmetic].’ (p. 117) No ontological commitment is given to the ideal agent; in this it is likened to an ideal gas. And for this reason we are able to ‘specify the capacities of the ideal agent by abstracting from the incidental limitations on our own collective practice’ (ibid.). The agent can do what we can do but can do it for collections however large, as we cannot. Thus modality is introduced without regard to human physical limitations. ‘Our geometrical statements can finally be understood as describing the performances of an ideal agent on ideal objects in an ideal space.’ (p. 124) Kitcher also alludes to the ‘double functioning of mathematical language—its use as a vehicle for the performance of mathematical operations as well as its reporting on those operations’ (p. 130). ‘To solve a problem is to discover a truth about mathematical operations, and to fiddle with the notation or to discern analogies in it is, on my account, to engage in those mathematical operations which one is attempting to characterize.’ (p. 131) The semiotic situation that Rotman discusses is brought up by Kitcher.

While Rotman is at pains to distinguish what he says from what Kitcher had written some years before the 1993 publication of *The ghost in Turing’s machine* [7] since he developed his theory independently and with different aims, we readers can regard his apparatus as a refinement of Kitcher’s since Rotman’s cast of characters includes an Agent to do the bidding of the character called the Subject. The Subject is Rotman’s idealization of the person that reads and writes mathematical text, and also the person that carries out some of the commands of the text. For example, it is the reader that obeys the command, ‘Consider triangle *ABC*.’ But it is the Agent (p. 73) that carries out such commands as, ‘Drop a perpendicular from vertex *A* to the line *BC*’, provided that the command is within the Agent’s capacities. We humans are well aware that we cannot draw straight lines; that is the work of the agents, Kitcher’s and Rotman’s. We reflect on the potential actions of these agents and address our reflections to other thinking Subjects. Rotman’s discussion of this is rich with details like the tenselessness of the commands to the Agent, indeed the complete lack of all indexicality in such texts. The tenselessness is an indication of why the Subject is an idealization as well as the Agent despite not being blessed with the supernatural powers of the Agent. The Agent, Rotman says, is like the person in a dream, the Subject like the person dreaming the dream, whereas in our normal state we real folk are more like the dreamer awake, what Rotman calls the Person to complete his semiotic hierarchy. Now Rotman transfers the whole enterprise to the texts so that mathematical statements are claims about what will result when certain operations are performed on signs (p. 77), and we need not follow him there to appreciate the serviceability of his semiotic distinctions.

The need for superhuman capacities was noted long ago in Frege’s ridicule [2] of the thought that mathematics is about empty symbols ([...] we would need an infinitely long blackboard, an infinite supply of chalk, and an infinite length of time—p. 199, §124). He also objected to a comparison to chess for Thomae’s formal theory of numbers while admitting that ‘there can be theorems in a *theory* of chess’ (p. 168, §93, my emphasis). According to Frege,

The distinction between the game itself and its theory, not drawn by Thomae, makes an essential contribution towards our understanding of the matter. [...] in the theory of chess it is not the chess pieces which are actually investigated; it is a question of the rules and their consequences. (pp. 168–169, §93)

### **The analogies between mathematics and games**

Having at our disposal the superhuman agents of Kitcher and Rotman, we are in a position to see what

is analogous in the matters of mathematics and games. It is not playing the game that is analogous to mathematics but our reflection in the role of subject on the playing of the game, which is done by the agent. When a column of figures is added up, we do it, and sometimes when the product of two elements of a group is required, we calculate it, but mathematics in the sense I am using here is not such mechanical processes at all but the investigation of their possibility, impossibility, and results. For that highly sophisticated reflective mathematical activity, the agent does the work because the agent can draw straight lines, and it is whether points are on the agent's straight lines that determines whether they are collinear, not whether they appear on the line in our sketch. We can put them on or off the line at will; the agent's results are constrained by the rules of the system in which the agent is working. Typically we have to deduce whether the agent's line is through a point or not. The agent, 'playing the game' according to the rules, gets the line through the point or not, but we have to figure it out. We *can* figure it out; the agent just does it. The analogy to games is two-fold.

1. The agent's mathematical activity (not playing a game) is analogous to the activity of playing a game like chess where it is clear what is possible and what is impossible—the same for every player—often superhuman but bound by rules. (Games like tennis depend for what is possible on physical skill, which is a red herring here.)

2. Our mathematical activity is analogous to (a) game invention and development, (b) the reflection on the playing of a game like chess that distinguishes expert play from novice play or (c) consideration of matters of play for their intrinsic interest apart from playing any particular match—merely human but not bound by rules.

It is we that deduce; the agent just does what it is told, provided that it is within the rules we have chosen. Analogous to the hypotheses of our theorems are chess positions, about which it is possible to reason as dependably as in mathematics because the structure is sufficiently precisely set out that everyone that knows the rules can see what statements about chess positions are legitimate and what are not. Chains of reasoning can be as long as we like without degenerating into the vagueness that plagues chains of reasoning about the real world. The ability to make and depend on such chains of reasoning in chess and other games is the ability that we need to make such chains in mathematics, as David Wells points out. To get a useful analogy here it is necessary to rise above the agent in the mathematics and the mere physical player in the game, but the useful analogy is dependent upon the positions in the game and the relations in the mathematics. The reflection in the game is about positions more than the play, and the mathematics is about relations and their possibility more than drawing circles or taking compact closures. Certainly the physical pieces used in chess and the symbols on paper are some distance below what is importantly going on.

I hope that the above makes clear why some rules are necessary to the analogy despite the fact that we are not bound by those rules. The rules are essential, since we could not do what we do without them, but it is the agent that is bound by them. We are talking about, as it were, what a particular choice of them does and does not allow. But our own activity is not bound by rules; we can say anything that conveys our meaning, anything that is convincing to others. Here is objectivity without objects. Chess reasoning is not dependent upon chess boards and chess men; it is dependent on the relations of positions mandated by the rules of the game of chess. Mathematics is not dependent on symbols (although they are as handy as chess sets) but on the relations of whatever we imagine the agent to work on, specified and reasoned about. Our conclusions are right or wrong as plainly as if we were ideal agents loose in Plato's heaven, but right or wrong dependent on what the axioms, conventions, or procedures we have chosen dictate. Outside mathematics, we reason routinely about what does not exist, most particularly about the future. As the novelist Jim Crace was quoted on page R10 of the 2007 6 2 Toronto *Globe and Mail*, 'As a good Darwinist, I know that what doesn't confer an advantage dies out. One advantage [of narrative (*Globe and Mail* addition)] is that it enables us to play out the bad things that might happen to us and to rehearse what we might do.' In order to tell our own stories, it is essential to project them hypothetically into the future based on observations and assumptions about the present. At its simplest and most certain, the skill involved is what allows one to note that if one moves this pawn forward one square the opponent's pawn can take it. It's about possibilities and of course impossibilities, all of them hypothetical. It is this fundamental skill that is used both in reflection on games and in mathematics to see what is necessary in their respective worlds.

I must make clear that David Wells thinks that entities in maths and abstract games have the same

epistemological status but that doing mathematics is like (an expert's) playing a game in several crucial respects, no more; he disagrees with the usefulness of bringing in ideal agents, indeed opposes doing so, apparently not seeing the advantage of splitting the analogy into the two numbered aspects above. This section is my attempt to outline a different but acceptable game analogy—a game-analysis analogy.

### Conclusion

Games like bridge and backgammon, which certainly involve strategy, have a stochastic element that prevents long chains of reasoning from being as useful as they are in chess. Such chains are, after all, an important part of how computers play chess. The probabilistic mathematics advocated by Doron Zeilberger [12] is analogous to the analysis of such a stochastic game and will be shunned by those uninterested in such analysis of something in which they see nothing stochastic. Classical (von Neumann) game theory, on the other hand actually is the analysis of situations that are called games and do involve strategy. The game theory of that current Princeton genius, John Conway, is likewise the actual analysis of game situations. [13] Does the existence of such mathematical analysis count for or against the general analogy between mathematics and game analysis? It would make an identification of mathematics with games one of those part-for-whole mistakes like 'all geometry is projective geometry' or 'arithmetic is just logic' from the nineteenth century, but identification is not the issue. It seems to me that its separation of game analysis from playing games tells in favour of the analogy of mathematics and analysis of games played by other—not *necessarily* superhuman—agents and against the analogy of mathematics and the expert play of the game itself. This is not a question David Wells has discussed. For him, an expert at abstract games like chess and go, play *is* expert play based unavoidably on analysis; analysis is just part of playing the game. Many are able to distinguish these activities, and not just hypothetically.

One occasionally hears the question, Is mathematics invented or discovered?—or an answer. As David Wells points out, even his game analogy shows why both answers and the answer 'both' are appropriate. Once a game is invented, the consequences are discovered—genuinely discovered, as it would require a divine intelligence to know how a game could best be played from new rules of any complexity. What happens in practice is that rules are adjusted so that the consequences may not be too drastically altered. Analogously, axioms are usually only adjusted and the altered consequences discovered.

What use can one make of this analogy? One use that one cannot make of it is as a stick to beat philosophers into admitting that mathematics is not problematic. Like mathematicians, they thrive on problems. Problems are the business of both mathematics and philosophy. Solving problems is the business of mathematics. If a philosopher came to regard the analogy as of some validity, then she would import into the hitherto unexamined territory of abstract games all of the philosophical problems to do with mathematics. Are chess positions real? How do we know about them? And so on; a new branch of philosophy would be invented.

What use then can mathematicians make of the analogy? We can use it as comparatively unproblematic material for use in discussing mathematics with those non-philosophers interested in understanding mathematics better. I have tried to indicate above some of the ways in which the analogy is both apt and of sufficient complexity to be interesting; it is no simple metaphor but can stand some exploration. Some of this exploration has been carried out by David Wells, to whose work I need to refer the reader.

## References

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