UNIVERSITY OF MANITOBA

DATE: December 18, 2009 — FINAL EXAMINATION
PAPER # 507 — TITLE PAGE
COURSE: MATH 1300 — TIME: 120 minutes
EXAMINATION: Vector Geometry and Linear Algebra — EXAMINER: Various

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FAMILY NAME: (Print in ink) ______________________________

GIVEN NAME(S): (Print in ink) ______________________________

STUDENT NUMBER: ____________

SIGNATURE: (in ink) ______________________________
(I understand that cheating is a serious offense)

- ❑ A01 Dr. Michael Doob [MWF 9:30-10:20]
- ❑ A02 Dr. Julien Arino [TR 8:30-9:45]
- ❑ A03 Dr. Michelle Davidson [MWF 11:30-12:20]
- ❑ A04 Dr. David Gunderson [MWF 13:30-14:20]
- ❑ A05 Dr. Julien Arino [TR 11:30-12:45]
- ❑ A91 Challenge for Credit
- ❑ Deferred examination

INSTRUCTIONS TO STUDENTS:

This is a 120 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 16 pages of questions, and one blank page (which may be removed). Please check that you have all the pages.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 80 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	20	
2	8	
3	6	
4	5	
5	13	
6	9	
7	4	
8	6	
9	9	
Total:	80	

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[3] 1. (a) Let V be a vector space, $W \subset V$. Under what conditions is W a subspace of V?

Solution: If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.

1. If u and v are vectors in W, then $u + v$ is in W.
   [W **closed under addition**]
2. If k is any scalar and u is any vector in W, then ku is in W.
   [W **closed under scalar multiplication**]

Marking scheme: Full marks for 1) Closed under addition and scalar multiplication and contains at least one element (for example, contains the zero vector) 2) Mathematical definition of properties 1 and 2 (since it states u and v are vectors in W, it assumes W contains elements).

[4] (b) Let A be an $n \times n$-matrix. State four additional properties equivalent to "A is invertible".

Solution: Four of the following:

1. $Ax = 0$ has only the trivial solution.
2. The reduced row-echelon form of A is I_n.
3. A can be expressed as a product of elementary matrices.
   [$A = E_1 \cdots E_n$, with $E_1, \ldots, E_n$ elementary matrices.]
4. $Ax = b$ is consistent for every matrix b.
   [$\forall b$, $Ax = b$ is consistent.]
5. $Ax = b$ has exactly one solution for every matrix b.
   [$\forall b$, $Ax = b$ has exactly one solution.]
6. $\det(A) \neq 0$.

Marking scheme: 1 per correct answer. If more than 4 answers provided, then the number of incorrect answers was deducted from 4. For example: 5 answers with 3 correct and 2 incorrect gave 2 marks.

[3] (c) Prove that $u \bullet (v \times w) = v \bullet (w \times u)$.

Solution: We have

$$\begin{aligned} u \bullet (v \times w) &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= - \begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= v \bullet (w \times u). \end{aligned}$$

Could also be done by computing explicitly the values of the scalar triple products in both cases.

Marking scheme: Setting up the problem: 1 mark. Starting computations: 1 mark. Getting the correct answer: 1 mark. Marks were subtracted for badly presented reasoning.

[2] (d) Prove the Cauchy-Schwarz inequality in 3-space.

Solution: The Cauchy-Schwarz inequality states that given two vectors u and v, the following inequality holds:

$$|u \bullet v| \leq \|u\| \, \|v\|.$$

In 3-space, the dot product of vectors u and v is defined by

$$u \bullet v = \|u\| \, \|v\| \cos\theta,$$

where θ is the angle between u and v. Therefore,

$$|u \bullet v| = \big| \|u\| \, \|v\| \cos\theta \big| = \|u\| \, \|v\| \, |\cos\theta| \leq \|u\| \, \|v\|$$

since $|\cos\theta| \leq 1$ for all θ, and the result is proved.

Marking scheme: Stating the result: 1 mark. Proving it: 1 mark.

[3] (e) State 3 of the axioms that must be satisfied for a set V to be a vector space.

Solution: Three of the following 10:

1. If u and v are in V, then $u + v$ is in V.
2. $u + v = v + u$.
3. $u + (v + w) = (u + v) + w$.
4. There is an object 0 in V, called a zero vector for V, such that $u + 0 = u$ for all u in V.
5. For each u in V, there is an object $-u$ in V, called a negative of u, such that $u + (-u) = (-u) + u = 0$.
6. If k is any scalar and u is any object in V, then ku is in V.
7. $k(u + v) = ku + kv$.
8. $(k + m)u = ku + mu$.
9. $k(mu) = (km)u$.
10. $1u = u$.

Marking scheme: 1 mark per correct answer. If more than 3 answers provided, then the number of incorrect answers was deducted from 3. For example: 5 answers with 3 correct and 2 incorrect gave 1 mark.

(f) Let $S = \{v_1, \dots, v_p\}$ be p vectors in an n-dimensional vector space V. State whether the following statements are True or False.
[Note: the score on question (f) is the number of right answers **minus** the number of wrong answers, with a minimum of 0.]

[1] i. If S is linearly independent, then $p = n$: **FALSE**

[1] ii. If S is a basis of V, then $p = n$: **TRUE**

[1] iii. If $\text{Span}(S) = W$, then W is a subspace of V: **TRUE**

[1] iv. If S is linearly independent, then S spans V: **FALSE**

[1] v. Let $S' = \{w_1, \dots, w_p\}$ be a basis of V. Then $S = S'$: **FALSE**

2. Let
$$W = \left\{ M \in \mathcal{M}_{22}; M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}, \text{ for all } a, b, d \in \mathbb{R} \right\}.$$

[4] (a) Show that W is a subspace of $\mathcal{M}_{22}$, the vector space of 2×2-matrices.

Solution: Let $k \in \mathbb{R}$,
$$M_1 = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix} \in W \text{ and } M_2 = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix} \in W.$$
Then
$$M_1 + M_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & d_1 + d_2 \end{bmatrix} \in W,$$
so W is closed under addition, and
$$kM_1 = \begin{bmatrix} ka_1 & kb_1 \\ 0 & kd_1 \end{bmatrix} \in W,$$
so W is closed under scalar multiplication. Therefore, W is a subspace of $\mathcal{M}_{22}$.

Marking scheme: 2 marks per condition. If $\in W$ is missing, take 1 mark off.

[4] (b) Is
$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$
a basis of W? If not, how should S be modified in order to be a basis of W?

Solution: S is a basis of W if $\text{span}(S) = W$ and S is linearly independent. We have
$$\begin{aligned}
\text{span}(S) &= \left\{ k_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + k_2 \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + k_3 \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} + k_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} k_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2k_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 2k_3 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k_4 \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} k_1 + 2k_2 & 2k_3 \\ 0 & k_4 \end{pmatrix} \right\},
\end{aligned}$$
for all $k_1, k_2, k_3, k_4 \in \mathbb{R}$. So $\text{span}(S) = W$.
On the other hand, we clearly have a relation between two of the matrices:
$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
and therefore S is not a basis of W since S is linearly dependent.
To get a basis, we use the Plus/Minus theorem:
$$\text{span}(S) = \text{span}\left(S - \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\} \right).$$
The set S' resulting from the cut,
$$S' = S - \left\{ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

is linearly independent and is a basis of W.

Marking scheme: 2 marks for establishing that S is not a basis because S is linearly dependent. 2 marks for finding a good way to work around the problem.

[6] 3. One of the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is expressible as a product of elementary matrices, the other is not. Express the one that is expressible as a product of elementary matrices as such, and explain why the other is not.

Solution: Matrix A has determinant 0. As a consequence, it cannot be expressed as a product of elementary matrices.

$|B| = 3$ so we know that B can be expressed as a product of elementary matrices. Perform, for instance, the following operations:

$$B \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix} \xrightarrow{R_2 \leftarrow -R_2/3} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - 2R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The elementary matrices E_1, E_2, E_3 and E_4 corresponding to the sequence of elementary operations indicated above allow to write $E_4E_3E_2E_1B = I$, and therefore $B = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$ and the matrices we seek are the matrices E_1^{-1}, E_2^{-1}, E_3^{-1} and E_4^{-1} that correspond to the reverse operations. Therefore,

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Marking scheme: 1 point for stating why A cannot be expressed as a product of elementary matrices. 2 marks for correctly performing elementary row operations on B. 1 mark for expressing B using the correct matrices. 2 marks for correctly expressing the inverse matrices.

Other expressions:

$$B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3/2 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

[5] 4. Is the vector $-1+9x-x^2$ in the span of the vectors

$$\{1+x+2x^3, 3x+x^2-x^3, 1-x+2x^2\}?$$

Explain.

Solution: Let us compute the span of the vectors (we call the set S for convenience):

$$\begin{aligned}\operatorname{span}(S) &= \{p(x) = k_1(1+x+2x^3)+k_2(3x+x^2-x^3)+k_3(1-x+2x^2)\}\\ &= \{p(x) = k_1+k_3+(k_1+3k_2-k_3)x+(k_2+2k_3)x^2+(2k_1-k_2)x^3\},\end{aligned}$$

for all $k_1, k_2, k_3 \in \mathbb{R}$. Therefore, the vector $-1+9x-x^2$ is in the span of S if we can find k_1, k_2, k_3 such that the coefficients of like degrees match, that is,

$$-1 = k_1+k_3, \quad 9 = k_1+3k_2-k_3, \quad -1 = k_2+2k_3 \text{ and } 0 = 2k_1-k_2.$$

We thus consider the system

$$k_1+k_3 = -1 \tag{1a}$$
$$k_1+3k_2-k_3 = 9 \tag{1b}$$
$$k_2+2k_3 = -1 \tag{1c}$$
$$2k_1-k_2 = 0 \tag{1d}$$

Therefore,

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & -1\\ 1 & 3 & -1 & 9\\ 0 & 1 & 2 & -1\\ 2 & -1 & 0 & 0\end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1\\ 0 & 3 & -2 & 10\\ 0 & 1 & 2 & -1\\ 0 & -1 & 2 & 2\end{array}\right]$$
$$\xrightarrow{R_4 \leftarrow R_4 + R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & -1\\ 0 & 3 & -2 & 10\\ 0 & 1 & 2 & -1\\ 0 & 0 & 0 & 1\end{array}\right].$$

This is a contradiction, so this system has no solution. So the vector $-1+9x-x^2$ is **not** in the span of the given vectors.

Marking scheme: 1 mark for writing the span properly. 1 mark for writing the system. 2 marks for rightfully concluding to the absence of solutions to the system, 1 for the conclusion that the vector does not belong to the span.

5. Let $P(1,2,3)$ and $Q(2,3,1)$ be two points in $\mathbb{R}^3$. The aim of this exercise is to find the coordinates of the points $R(a,b,c) \in \mathbb{R}^3$ such that

$$\overrightarrow{PQ} \perp \overrightarrow{PR} \quad \text{and} \quad \|\overrightarrow{QR}\| = 3, \tag{*}$$

where the symbol $\perp$ means "orthogonal to".

[2] (a) Find an equation that must be satisfied by the coordinates a, b, c of R in order that $\overrightarrow{PQ} \perp \overrightarrow{PR}$.

Solution: We have $\overrightarrow{PQ} = (1,1,-2)$ and $\overrightarrow{PR} = (a-1, b-2, c-3)$. Then

$$\begin{aligned} \overrightarrow{PQ} \perp \overrightarrow{PR} &\Leftrightarrow \overrightarrow{PQ} \bullet \overrightarrow{PR} = 0 \\ &\Leftrightarrow (a-1) + (b-2) - 2(c-3) = 0 \\ &\Leftrightarrow a + b - 2c + 3 = 0. \end{aligned}$$

Marking scheme: Clearly stating that $\overrightarrow{PQ} \perp \overrightarrow{PR} \Leftrightarrow \overrightarrow{PQ} \bullet \overrightarrow{PR} = 0$: 1 mark. Obtaining the correct equation for the plane (whatever the form): 1 mark.

[1] (b) What is the nature of the set of values of a, b, c satisfying the condition found in (a)?

Solution: The set

$$\{(a,b,c) \in \mathbb{R}^3; \quad a + b - 2c + 3 = 0\}$$

is a plane in $\mathbb{R}^3$.

[2] (c) State the Pythagorean Theorem in n-space.

Solution: Let u and v be two orthogonal vectors in n-space, then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2.$$

Marking scheme: Stating some form of the theorem (even in two-space) but forgetting to indicate that the vectors must be orthogonal: 1 mark. Correct statement gets full marks.

[2] (d) Compute the norms of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

Solution:

$$\|\overrightarrow{PQ}\| = \sqrt{6} \text{ and } \|\overrightarrow{PR}\| = \sqrt{(a-1)^2 + (b-2)^2 + (c-3)^2}$$

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[2] (e) Using the Pythagorean Theorem, find an equation that must be satisfied by the coordinates a, b, c of R. [Hint: this equation is not linear.]

Solution: Since $\overrightarrow{PQ} \perp \overrightarrow{PR}$, we can use the Pythagorean theorem:

$$\|\overrightarrow{PQ}\|^2 + \|\overrightarrow{PR}\|^2 = \|\overrightarrow{PQ} + \overrightarrow{PR}\|^2.$$

The triangle PQR is a right triangle with hypothenuse QR, and a simple geometric argument shows that $\|\overrightarrow{PQ} + \overrightarrow{PR}\| = \|\overrightarrow{QR}\|$: let S be the point defined by

$$S = P + \overrightarrow{PQ} + \overrightarrow{PR},$$

then $PQSR$ is a rectangle with diagonals PS and QR. Therefore, the conclusion of the Pythagorean theorem can be rewritten as

$$\|\overrightarrow{PQ}\|^2 + \|\overrightarrow{PR}\|^2 = \|\overrightarrow{QR}\|^2.$$

Using the norms computed earlier and the fact that $\|QR\| = 3$ by assumption, we obtain

$$6 + (a-1)^2 + (b-2)^2 + (c-3)^2 = 9,$$

i.e.,

$$(a-1)^2 + (b-2)^2 + (c-3)^2 = 3.$$

Marking scheme: Writing down the Pythagorean theorem in the case of the exercise: 1 mark. Deducing some (correct) form of the equation: 1 mark. 1 mark was also given to some incorrect solutions (not using the Pythagorean theorem) that *clearly* indicated a norm related equation, e.g., deducing an equation from $\|\overrightarrow{QR}\| = 3$.

[1] (f) What is the nature of the set of values of a, b, c satisfying the conditions found in (e)? [You can use a geometrical argument.]

Solution: The set

$$\{(a, b, c) \in \mathbb{R}^3; \quad (a-1)^2 + (b-2)^2 + (c-3)^2 = 3\}$$

is the sphere with radius $\sqrt{3}$ and centered at $(1, 2, 3)$, i.e., at P.

Marking scheme: Although not correct, stating that the equation is a quadratic was accepted.

[2] (g) Deduce a system characterizing the set of points satisfying (*).

Solution: Put together the answers to (a) and (e):

$$\begin{aligned} a+b-2c &= -3 \\ (a-1)^2+(b-2)^2+(c-3)^2 &= 3. \end{aligned}$$

This was sufficient. But we could go further (bonus mark): expand the second equation, giving $a^2-2a+1+b^2-4b+4+c^2-6c+9=3$. Rewrite as $a^2+b^2+c^2-2(a+b-2c)-2b-10c+14=3$. Using the first equation, $a^2+b^2+c^2-2b-10c+20=3$. Collecting, $a^2+(b-1)^2+(c-5)^2-6=3$, so finally, the system reduces to $a^2+(b-1)^2+(c-5)^2=3^2$.

[1] (h) What is the nature of the set of values of a, b, c satisfying the conditions found in (g)?

Solution: The intersection of a plane and a sphere: a circle.

Marking scheme: Although not correct, stating that the equation is a quadratic was accepted.

[3] 6. (a) Find the point of intersection of the lines L_1 and L_2 defined by

$$\begin{aligned} L_1 &: (x, y, z) = (1, 2, 1) + t(2, 1, 2), \quad t \in \mathbb{R} \\ L_2 &: (x, y, z) = (2, 1, 2) + s(1, 2, 1), \quad s \in \mathbb{R}. \end{aligned}$$

Solution: By inspection, $s = t = 1$ gives the common point $(3, 3, 3)$. If inspection is not used, comparing coordinates,

$$\left| \begin{array}{lcl} x & : & 1 + 2t = 2 + s \\ y & : & 2 + t = 1 + 2s \\ z & : & 1 + 2t = 2 + s \end{array} \right.$$

$$\begin{aligned} &\Rightarrow s = 2t - 1 \\ &\Rightarrow 2 + t = 1 + 2(2t - 1) \\ &\Rightarrow t = 1 \\ &\Rightarrow s = 1. \end{aligned}$$

So in either equations, $s = 1 = t$ gives the point of intersection $(x, y, z) = (3, 3, 3)$.

Marking scheme: 2 points for the right idea, faulty values of s, t. Solving the system using row operations is also good.

[3] (b) Find an equation of the plane containing the two lines L_1 and L_2.

Solution: The plane desired has normal n orthogonal to L_1 and L_2, so one can use

$$n = (2, 1, 2) \times (1, 2, 1) = (-3, 0, 3).$$

Using, say, the point $(1, 2, 1)$ on L_1, the equation is

$$(-3, 0, 3) \bullet (x - 1, y - 2, z - 1) = 0.$$

Simplifying,

$$3x - 3z = 0$$

or

$$x = z.$$

Options: use $(2, 1, 2)$ from L_2 or $(3, 3, 3)$ from part (a).

Marking scheme: 3 marks for getting to $3x - 3z = 0$.

[3] (c) Find the point on L_1 closest to the point $(3, 2, 0)$.

Solution: Option 1. Let R be the desired point, $Q = (3, 2, 0)$ and $v = (2, 1, 2)$. Pick $P = (1, 2, 1)$ on L_1. Then $\overrightarrow{PQ} = (2, 0, -1)$. Then

$$\overrightarrow{PR} = \mathrm{proj}_v \overrightarrow{PQ} = \frac{\overrightarrow{PQ}}{\|v\|^2} v = \frac{2}{9}(2, 1, 2).$$

So

$$R = P + \overrightarrow{PR} = (1, 2, 1) + \frac{2}{9}(2, 1, 2) = \left(\frac{13}{9}, \frac{20}{9}, \frac{13}{9}\right).$$

Option 2. Same as above but with $P = (3, 3, 3)$ from (a).

Option 3. Let $Q = (3, 2, 0)$, $v = (2, 1, 2)$ and let R be the desired point on L_1. Then both Q and R lie on a plane with normal v. Writing the point normal form of the equation of that plane,

$$2(x - 3) + (y - 2) + 2(z - 0) = 0,$$

that is, $2x + y + 2z = 8$. Now use the parametric equation for L_1, find $t = 2/9$ and get R.

7. Let $P(1,2,3)$ be a point and $(x,y,z) = (4,1,2) + t(1,2,-1)$, $t \in \mathbb{R}$, be the equation of a line L in $\mathbb{R}^3$.

[2] (a) Find the equation of the plane through P perpendicular to the line L.

Solution: L plays the role of a direction normal to the plane, and thus we take the normal $(1,2,-1)$. The point-normal form of the equation of the plane is

$$(1,2,-1) \bullet (x-1, y-2, z-3) = 0,$$

or, in other words,

$$x + 2y - z - 2 = 0.$$

[2] (b) Find the equation of a line through P perpendicular to the plane with equation $2x - y - z + 4 = 0$.

Solution: A normal to the plane is $n = (2,-1,-1)$, and thus an equation of the line is

$$(x,y,z) = (1,2,3) + t(2,-1,-1), \quad t \in \mathbb{R}.$$

8. Consider the following system of linear equations:

$$\begin{aligned} x+y+z &= 1 \\ 2x+y+z &= 2 \\ 3x+ay+bz &= c. \end{aligned}$$

For what values of a, b and c does this system have

[2] (a) no solutions

[2] (b) one solution

[2] (c) more than one solution

Solution: Write the matrix associated to the system:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & a & b \end{bmatrix}.$$

We have

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 3 & b \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 3 & a \end{vmatrix} \\ &= b-a-2b+3+2a-3 \\ &= a-b. \end{aligned}$$

So the system has a unique solution if $a \neq b$ and for any c.

Suppose now that $a = b$, and write the augmented matrix corresponding to the system:

$$M = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 \\ 3 & a & a & c \end{array}\right].$$

Perform elementary operations:

$$\begin{aligned} M &\xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & a-3 & a-3 & c-3 \end{array}\right] \\ &\xrightarrow{R_3 \leftarrow R_3 + (a-3)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & c-3 \end{array}\right] \end{aligned}$$

Therefore, if $c = 3$, the system has infinitely many solutions, and if $c \neq 3$, it has none. To summarize:

(a) If $a = b$ and $c \neq 3$, then the system has **no solution**.

(b) If $a \neq b$, then for any c the system has a **unique solution**.

(c) If $a = b$ and $c = 3$, then the system has **infinitely many solutions**.

9. The matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 2 \\ 3 & 4 & 5 & 2 & 4 \\ 1 & 3 & 0 & -1 & -2 \\ 0 & -1 & 1 & 1 & 2 \end{bmatrix}$$

has reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 3 & 2 & 4 \\ 0 & 1 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[1] (a) The dimension of the row space of A is **2**

[2] (b) A basis for the row space of A is:

Solution:

$$\{(1, 0, 3, 2, 4), (0, 1, -1, -1, -2)\}.$$

Marking scheme: The vectors must be listed as above. Or it must be clear that you are listing the vectors that make up a basis. Vectors listed not in a set and with no explanation: 1 mark only. Listing correct vectors but as column vectors: 1 mark.

[1] (c) The dimension of the column space of A is **2**

[2] (d) A basis for the column space of A is:

Solution:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \\ -1 \end{bmatrix} \right\}.$$

Marking scheme: The vectors must be listed as above. Or it must be clear that you are listing the vectors that make up a basis. Vectors listed not in a set and with no explanation: 1 mark only. Listing correct vectors but as row vectors: 1 mark. Using vectors from the reduced row echelon form of A and not A: 0 marks.

[1] (e) The dimension of the null space of A is **3**

[2] (f) A basis for the null space of A is:

Solution: The solution to $Ax = 0$ takes the form

$$(x_1, x_2, x_3, x_4, x_5) = (-3r - 2s - 4t, r + s + 2t, r, s, t),$$

or, if written vertically,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3r - 2s - 4t \\ r + s + 2t \\ r \\ s \\ t \end{bmatrix}$$

for $r, s, t \in \mathbb{R}$. Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and a basis for the nullspace is

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Marking scheme: Both row and column forms of the vectors were accepted here. Stating the general form of the solution: 1 mark. Stating the vectors: 1 mark.