Question. *Show that $\mathbb{R}^2$ is a vector space.*

Solution. We need to check each and every axiom of a vector space to know that it is in fact a vector space.

$A1$: Let $(a, b), (c, d) \in \mathbb{R}^2$. Then

$$(a, b) + (c, d) = (a + c, b + d) \in \mathbb{R}^2.$$

Therefore $A1$ holds.

$A2$: Let $(a, b), (c, d) \in \mathbb{R}^2$. Then

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ &= (c + a, d + b)\\ &= (c, d) + (a, b).\end{aligned}$$

Therefore $A2$ holds.

$A3$: Let $(a, b), (c, d), (e, f) \in \mathbb{R}^2$. Then

$$\begin{aligned}(a, b) + ((c, d) + (e, f)) &= (a, b) + (c + e, d + f)\\ &= (a + (c + e), b + (d + f))\\ &= ((a + c) + e, (b + d) + f)\\ &= (a + c, b + d) + (e, f)\\ &= ((a, b) + (c, d)) + (e, f).\end{aligned}$$

Therefore $A3$ holds.

$A4$: Our claim is that the vector $(0, 0)$ works. Let $(a, b) \in \mathbb{R}^2$. Then

$$\begin{aligned}(0, 0) + (a, b) &= (0 + a, 0 + b)\\ &= (a, b)\end{aligned}$$

Therefore, $A4$ holds.

$A5$: Let $(a, b) \in \mathbb{R}^2$. Then we need to find "$-(a, b)$". Our claim is that $(-a, -b)$ works.

$$\begin{aligned}(a, b) + (-a, -b) &= (a + -a, b + -b)\\ &= (0, 0) = \mathbf{0}.\end{aligned}$$

$M1$: Let $k \in \mathbb{R}, (a, b) \in \mathbb{R}^2$. Then

$$k(a, b) = (ka, kb) \in \mathbb{R}^2.$$

Therefore $M1$ holds.

$M2$: Let $k \in \mathbb{R}$, $(a, b), (c, d) \in \mathbb{R}^2$. Then,

$$\begin{aligned}k((a, b) + (c, d)) &= k(a + c, b + d)\\ &= (k(a + c), k(b + d))\\ &= (ka + kc, kb + kd)\\ &= (ka, kb) + (kc, kd)\\ &= k(a, b) + k(c, d).\end{aligned}$$

Therefore $M2$ holds.

$M3$: Let $k, m \in \mathbb{R}$, $(a,b) \in \mathbb{R}^2$. Then

$$\begin{aligned}(k+m)(a,b) &= ((k+m)a, (k+m)b)\\ &= (ka+ma, kb+mb)\\ &= (ka, kb) + (ma, mb)\\ &= k(a,b) + m(a,b).\end{aligned}$$

Therefore $M3$ holds.

$M4$: Let $k, m \in \mathbb{R}$, $(a,b) \in \mathbb{R}^2$. Then

$$\begin{aligned}k(m(a,b)) &= k(ma, mb)\\ &= (k(ma), k(mb))\\ &= ((km)a, (km)b)\\ &= (km)(a,b).\end{aligned}$$

Therefore $M4$ holds.

$M5$: Let $(a,b) \in \mathbb{R}^2$. Then,

$$1(a,b) = (1a, 1b) = (a,b).$$

Therefore $M5$ holds.

Question. *Show that $M_{2,2}$, the set of all 2×2 matrices, is a vector space.*

Solution. We need to check each and every axiom of a vector space to know that it is in fact a vector space.

$A1$: Let $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \in M_{2,2}$. Then

$$\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} \end{bmatrix},$$

which is also a 2×2 matrix. Therefore $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \in M_{2,2}$.

$A2$: Let $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}, \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} \in M_{2,2}$. Then

$$\begin{aligned}
\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \left(\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} + \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} \right) &= \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \left(\begin{bmatrix} b_{1,1}+c_{1,1} & b_{1,2}+c_{1,2} \\ b_{2,1}+c_{2,1} & b_{2,2}+c_{2,2} \end{bmatrix} \right) \\
&= \begin{bmatrix} a_{1,1}+b_{1,1}+c_{1,1} & a_{1,2}+b_{1,2}+c_{1,2} \\ a_{2,1}+b_{2,1}+c_{2,1} & a_{2,2}+b_{2,2}+c_{2,2} \end{bmatrix} \\
&= \left(\begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} \end{bmatrix} \right) + \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} \\
&= \left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \right) + \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix}.
\end{aligned}$$

$A3$: Let $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \in M_{2,2}$. Then

$$\begin{aligned}
\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} &= \begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} \end{bmatrix} \\
&= \begin{bmatrix} b_{1,1}+a_{1,1} & b_{1,2}+a_{1,2} \\ b_{2,1}+a_{2,1} & b_{2,2}+a_{2,2} \end{bmatrix} \\
&= \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} + \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.
\end{aligned}$$

$A4$: The vector $\mathbf{0}$ is the zero matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, since for any $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in M_{2,2}$,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

$A5$: Let $A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in M_{2,2}$, and define $-A$ to be the matrix $\begin{bmatrix} -a_{1,1} & -a_{1,2} \\ -a_{2,1} & -a_{2,2} \end{bmatrix}$. Then

$$\begin{aligned}
\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} -a_{1,1} & -a_{1,2} \\ -a_{2,1} & -a_{2,2} \end{bmatrix} &= \begin{bmatrix} a_{1,1}-a_{1,1} & a_{1,2}-a_{1,2} \\ a_{2,1}-a_{2,1} & a_{2,2}-a_{2,2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

which is the zero vector $\mathbf{0}$ as required.

$M1$: Let $r \in \mathbb{R}$, and let $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in M_{2,2}$. Let

$$r \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} ra_{1,1} & ra_{1,2} \\ ra_{2,1} & ra_{2,2} \end{bmatrix} \in M_{2,2}.$$

$M2$: Let $r \in \mathbb{R}$, and let $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}, \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} \in M_{2,2}$. Then

$$\begin{aligned} r\left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}\right) &= r\left(\begin{bmatrix} a_{1,1}+b_{1,1} & a_{1,2}+b_{1,2} \\ a_{2,1}+b_{2,1} & a_{2,2}+b_{2,2} \end{bmatrix}\right) \\ &= \begin{bmatrix} r(a_{1,1}+b_{1,1}) & r(a_{1,2}+b_{1,2}) \\ r(a_{2,1}+b_{2,1}) & r(a_{2,2}+b_{2,2}) \end{bmatrix} \\ &= \begin{bmatrix} ra_{1,1}+rb_{1,1} & ra_{1,2}+rb_{1,2} \\ ra_{2,1}+rb_{2,1} & ra_{2,2}+rb_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} ra_{1,1} & ra_{1,2} \\ ra_{2,1} & ra_{2,2} \end{bmatrix} + \begin{bmatrix} rb_{1,1} & rb_{1,2} \\ rb_{2,1} & rb_{2,2} \end{bmatrix}. \end{aligned}$$

$M3$: Let $r, s \in \mathbb{R}$, and let $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in M_{2,2}$. Then

$$\begin{aligned} (r+s)\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} &= \begin{bmatrix} (r+s)a_{1,1} & (r+s)a_{1,2} \\ (r+s)a_{2,1} & (r+s)a_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} ra_{1,1}+sa_{1,1} & ra_{1,2}+sa_{1,2} \\ ra_{2,1}+sa_{2,1} & ra_{2,2}+sa_{2,2} \end{bmatrix} \\ &= \begin{bmatrix} ra_{1,1} & ra_{1,2} \\ ra_{2,1} & ra_{2,2} \end{bmatrix} + \begin{bmatrix} sa_{1,1} & sa_{1,2} \\ sa_{2,1} & sa_{2,2} \end{bmatrix}. \end{aligned}$$

$M4$: Let $r, s \in \mathbb{R}$, and let $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in M_{2,2}$. Then

$$\begin{aligned} r\left(s\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}\right) &= r\left(\begin{bmatrix} sa_{1,1} & sa_{1,2} \\ sa_{2,1} & sa_{2,2} \end{bmatrix}\right) \\ &= \begin{bmatrix} r(sa_{1,1}) & r(sa_{1,2}) \\ r(sa_{2,1}) & r(sa_{2,2}) \end{bmatrix} \\ &= \begin{bmatrix} (rs)a_{1,1} & (rs)a_{1,2} \\ (rs)a_{2,1} & (rs)a_{2,2} \end{bmatrix} \\ &= (rs)\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \end{aligned}$$

$M5$: For any $\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \in M_{2,2}$,

$$1\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} 1a_{1,1} & 1a_{1,2} \\ 1a_{2,1} & 1a_{2,2} \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

Question. *Determine if the set V of solutions of the equation $2x - 3y + z = 1$ is a vector space or not. Determine which axioms of a vector space hold, and which ones fail.*

The set V (together with the standard addition and scalar multiplication) is **not** a vector space. In fact, many of the rules that a vector space must satisfy do not hold in this set. What follows are all the rules, and either proofs that they do hold, or counter examples showing they do not hold.

$A1$: $\mathbf{u}, \mathbf{v} \in V \implies \mathbf{u} + \mathbf{v} \in V$ (closure under addition)

Let $\mathbf{u}, \mathbf{v} \in V$. That is, $\mathbf{u}$ and $\mathbf{v}$ are solutions to the equation $2x - 3y + z = 1$. More specifically, $\mathbf{u}$ and $\mathbf{v}$ are triples in $\mathbb{R}^3$, say $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, such that $2u_1 - 3u_2 + u_3 = 1$ and $2v_1 - 3v_2 + v_3 = 1$.

$A1$ does not hold here. For instance, take $\mathbf{u} = (0, 0, 1)$ and $\mathbf{v} = (1, 0, -1)$ (both are in V since both are solutions to $2x - 3y + z = 1$). Then $\mathbf{u} + \mathbf{v} = (1, 0, 0)$, but $2(1) - 3(0) + 0 = 2 \neq 1$, and therefore $\mathbf{u} + \mathbf{v}$ is not a solution to $2x - 3y + z = 1$, showing that $\mathbf{u} + \mathbf{v} \notin V$.

$A2$: Associativity holds since the real numbers are associative. Let $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3), \mathbf{w} = (w_1, w_2, w_3) \in V$. Then

$$\begin{aligned}
\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (u_1, u_2, u_3) + ((v_1, v_2, v_3) + (w_1, w_2, w_3)) \\
&= (u_1, u_2, u_3) + (v_1 + w_1, v_2 + w_2, v_3 + w_3) \\
&= (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3) \\
&= (u_1 + v_1, u_2 + v_2, u_3 + v_3) + (w_1, w_2, w_3) \\
&= ((u_1, u_2, u_3) + (v_1, v_2, v_3)) + (w_1, w_2, w_3) \\
&= (\mathbf{u} + \mathbf{v}) + \mathbf{w}
\end{aligned}$$

$A3$: Commutativity holds (similar to associativity above) just since the real numbers are commutative. Let $\mathbf{u} = (u_1, u_2, u_3), \mathbf{v} = (v_1, v_2, v_3) \in V$. Then

$$\begin{aligned}
\mathbf{u} + \mathbf{v} &= (u_1, u_2, u_3) + (v_1, v_2, v_3) \\
&= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\
&= (v_1 + u_1, v_2 + u_2, v_3 + u_3) \\
&= (v_1, v_2, v_3) + (u_1, u_2, u_3) \\
&= \mathbf{v} + \mathbf{u}
\end{aligned}$$

$A4$: The requirement of a zero vector fails since there is only one possibility for a zero vector using the standard addition: the vector $(0, 0, 0)$, which is not a solution to the equation $2x - 3y + z = 1$ (since $2(0) - 3(0) + (0) = 0$). Therefore, $A4$ fails.

$A5$: The requirement of additive inverses fails as well. For instance, $(0, 0, 1)$ is an element of V (as mentioned in $A1$ above). The additive inverse of $(0, 0, 1)$, using the standard addition, would be $(0, 0, -1)$. However, $(0, 0, -1)$ is not a solution to the equation $2x - 3y + z = 1$. Therefore $(0, 0, -1) \notin V$. In fact, the following proposition shows that for every $\mathbf{v} \in V$, $-\mathbf{v} \notin V$:

Proposition. *If $\boldsymbol{v} \in V$, then $-\boldsymbol{v} \notin V$.*

Proof. Let $\mathbf{v} = (v_1, v_2, v_3) \in V$. Then $\mathbf{v}$ is a solution to the equation $2x - 3y + z = 1$, that is, $2v_1 - 3v_2 + v_3 = 1$.

Plugging in $-\mathbf{v} = (-v_1, -v_2, -v_3)$ into the equation, we get:

$$\begin{aligned}
2(-v_1) - 3(-v_2) + (-v_3) &= -2v_1 + 3(v_2) - v_3 \\
&= (-1)(2v_1 - 3v_2 + v_3) \\
&= (-1)(1) && \text{(since } 2v_1 - 3v_2 + v_3 = 1\text{)} \\
&= -1 \\
&\neq 1.
\end{aligned}$$

Therefore, $-\mathbf{v} = (-v_1, -v_2, -v_3)$ is not a solution to $2x - 3y + z = 1$, and is therefore not in V. $\square$

One might say the above proposition proves this space TOTALLY fails $A5$, since for EVERY $\mathbf{v} \in V$, the additive inverse of $\mathbf{v}$, that is $-\mathbf{v}$, is not in V. Note the above also produces an infinite number of counter examples to $M1$ (below). For every $\mathbf{v} \in V$, $(-1)\mathbf{v} \notin V$.

$M1$: Closure under multiplication does not hold. For example, take $2 \in \mathbb{R}$, and $(0, 0, 1) \in V$. Then $2(0, 0, 1) = (0, 0, 2)$, but $(0, 0, 2)$ is not a solution to the equation $2x - 3y + z = 1$ (since $2(0) - 3(0) + (2) = 2 \neq 1$). Therefore, the space is not closed under scalar multiplication.

$M2$, $M3$, $M4$, and $M5$ all hold, for the same reasons they hold in $\mathbb{R}^3$ with the standard addition, proofs similar to the above for $A2$ and $A3$.

Question. *Let V be the set $V = \{red, blue\}$, that is, V is the finite set consisting of just the two elements "red" and "blue". Define an addition on this set as follows:*

$\oplus$	*Red*	*Blue*
Red	*Red*	*Blue*
Blue	*Blue*	*Red*

For any real number r, for any $\boldsymbol{v} \in V$, define the scalar multiplication $r \cdot \boldsymbol{v}$ as

$$r \cdot \boldsymbol{v} = \begin{cases} red & \text{if } r \geq 0, \\ blue & \text{otherwise.} \end{cases}$$

Show which axioms of a vector space hold, and which fail for this set V together with these operations $\cdot$ and $\oplus$.

Solution.

$A1$: Let $\mathbf{u}, \mathbf{v} \in V$. Then by the definition of the addition, $\mathbf{u} + \mathbf{v}$ is either red or blue, and is thus again in V. Therefore V is closed under addition ($A1$ holds).

$A2$: Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$. To show this holds, we could check every possibility for $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ one by one, but this would take a while. To do it faster, I'll be a little smarter about what I check. If $\mathbf{u} = \text{red}$, then

$$\begin{aligned} \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) &= \text{red} \oplus (\mathbf{v} \oplus \mathbf{w}) \\ &= \mathbf{v} \oplus \mathbf{w} && (\text{red} \oplus \mathbf{x} = \mathbf{x}) \\ &= (\text{red} \oplus \mathbf{v}) \oplus \mathbf{w} && (\text{red} \oplus \mathbf{v} = \mathbf{v}) \\ &= (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}, \end{aligned}$$

which is what was needed to be shown.

If $\mathbf{v} = \text{red}$, then similarly,

$$\begin{aligned} \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) &= \mathbf{u} \oplus (\text{red} \oplus \mathbf{w}) \\ &= \mathbf{u} \oplus \mathbf{w} \\ &= (\mathbf{u} \oplus \text{red}) \oplus \mathbf{w} && (\mathbf{u} \oplus \text{red} = \mathbf{u}) \\ &= (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}, \end{aligned}$$

which is what was needed to be shown.

If $\mathbf{w} = \text{red}$, then,

$$\begin{aligned} \mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) &= \mathbf{u} \oplus (\mathbf{v} \oplus \text{red}) \\ &= \mathbf{u} \oplus \mathbf{v} \\ &= (\mathbf{u} \oplus \mathbf{v}) \oplus \text{red} \\ &= (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}, \end{aligned}$$

which again is what was needed to be shown.

If none of them are red, then all of them are blue, and we get:

$$\begin{aligned} \text{blue} \oplus (\text{blue} \oplus \text{blue}) &= \text{blue} \oplus \text{red} \\ &= \text{blue} \\ &= \text{red} \oplus \text{blue} \\ &= (\text{blue} \oplus \text{blue}) \oplus \text{blue}, \end{aligned}$$

which yet again is what needed to be shown. Therefore, $A2$ holds.

$A3$: Let $\mathbf{u}, \mathbf{v} \in V$. If $\mathbf{u} = \mathbf{v}$, then clearly $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$. If $\mathbf{u} \neq \mathbf{v}$, then one is red and the other is blue, and since red $\oplus$ blue = blue $\oplus$ red (= blue), we have that $A3$ holds.

$A4$: We have already sort of seen that there is a zero vector. The zero vector here is red. One can check this quickly:

$$\begin{aligned} red \oplus red &= red \\ blue \oplus red &= blue \end{aligned}$$

Therefore, for all $\mathbf{v} \in V$, $\mathbf{v} + red = \mathbf{v}$. Therefore red is a zero vector, and $A4$ holds.

$A5$: To show that $A5$ holds, it suffices to find, "-red" and "-blue". The element "-red" (if it exists) should have the property that red $\oplus$ (-red) equals the zero element, which is also red. Note that red $\oplus$ red = red. Therefore choosing -red = red works.

Similarly, "-blue" is an element (if any exists) such that blue $\oplus$ (-blue) equals the zero element, red. Note that blue $\oplus$ blue = red. Therefore -blue = blue works. Therefore $A5$ holds.

$M1$: Let $k \in \mathbb{R}$, $\mathbf{v} \in V$. Then $k \cdot \mathbf{v}$ is, by the definition, either red or blue, and is thus in V. Therefore $M1$ holds.

$M2$: Let $k \in \mathbb{R}$, and let $\mathbf{u}, \mathbf{v} \in V$. Then

$$k \cdot (\mathbf{u} \oplus \mathbf{v}) = \begin{cases} red \text{ if } k \geq 0, \\ blue \text{ otherwise.} \end{cases}$$

$$\begin{aligned} k \cdot \mathbf{u} \oplus k \cdot \mathbf{v} &= \begin{cases} red \oplus red \text{ if } k \geq 0 \\ blue \oplus blue \text{ otherwise} \end{cases} \\ &= red. \end{aligned}$$

This gives us a hint which values to pick to find a counter example. Pick $k = -1$, and $\mathbf{v} = \mathbf{u} = \text{red}$. Then,

$$\begin{aligned} -1 \cdot (\text{red} \oplus \text{red}) &= -1 \cdot \text{red} \\ &= \text{blue}, \end{aligned}$$

but

$$\begin{aligned} -1 \cdot \text{red} \oplus -1 \cdot \text{red} &= \text{blue} \oplus \text{blue} \\ &= \text{red}. \end{aligned}$$

Therefore, $M2$ fails for the specific case $k = -1$ and $\mathbf{u} = \mathbf{v} = \text{red}$, and so $M2$ fails.

$M3$: I claim that $M3$ fails. To show this, I need to produce two real numbers r and s, and a vector $\mathbf{v} \in V$ such that $(r + s) \cdot \mathbf{v} \neq r \cdot \mathbf{v} + s \cdot \mathbf{v}$.

Pick $r = -1$, $s = 2$, and $\mathbf{v} = \text{red}$. Then,

$$\begin{aligned}(-1 + 2) \cdot \text{red} &= 1 \cdot \text{red} \\ &= \text{red},\end{aligned}$$

but

$$\begin{aligned}-1 \cdot \text{red} + 2 \cdot \text{red} &= \text{blue} + \text{red} \\ &= \text{blue}.\end{aligned}$$

Therefore $M3$ fails.

$M4$: My claim is that $M4$ fails. To see this, let $r = 2$, $s = -2$, and $\mathbf{v} = \text{red}$. Then,

$$\begin{aligned}2 \cdot (-2 \cdot red) &= 2 \cdot blue \\ &= red,\end{aligned}$$

but

$$\begin{aligned}(2 \times -2) \cdot red &= -4 \cdot red \\ &= blue.\end{aligned}$$

Therefore $M4$ fails.

$M5$: This fails since for <u>any</u> $\mathbf{v} \in V$, $1 \cdot \mathbf{v} = \text{red}$. Specifically,

$$1 \cdot blue = red,$$

and therefore $M5$ fails.

Therefore, $A1$, $A2$, $A3$, $A4$, $A5$, and $M1$ all hold, while $M2$, $M3$, and $M4$ all fail. Therefore this is not a vector space.

Question. *Let V be the set $\mathbb{R}^2$ with the following operations defined as follows:*

- *for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, define*
$$(x_1, y_1) + (x_2, y_2) = (2(x_1 + y_1 + x_2 + y_2), -1(x_1 + y_1 + x_2 + y_2)).$$
- *for any $k \in \mathbb{R}$, and for any $(x_1, y_1) \in \mathbb{R}^2$, define $k(x_1, y_1) = (kx_1, ky_1)$.*

Is V together with these operations a vector space? If so, prove it. If not, show all axioms that fail, and explain why they fail.

Solution.

To check that V is a vector space, one must check each of the 10 axioms of a vector space to see if they hold.

$A1$: Let $(a, b), (c, d) \in V$. Then

$$(a, b) + (c, d) = (2(a + b + c + d), -1(a + b + c + d)) \in V.$$

Therefore V is closed under addition ($A1$ holds).

$A2$: Let $(a, b), (c, d), (e, f) \in V$. Then

$$\begin{aligned}(a, b) + ((c, d) + (e, f)) &= (a, b) + (2(c + d + e + f), -1(c + d + e + f)) \\ &= (2(a + b + 2(c + d + e + f) + -1(c + d + e + f)), -1(a + b + 2(c + d + e + f) + -1(c + d + e + f))) \\ &= (2(a + b + c + d + e + f), -1(a + b + c + d + e + f))\end{aligned}$$

$$\begin{aligned}((a, b) + (c, d)) + (e, f) &= (2(a + b + c + d), -1(a + b + c + d)) + (e, f) \\ &= (2(2(a + b + c + d) + -1(a + b + c + d) + e + f), -1(2(a + b + c + d) + -1(a + b + c + d) + e + f)) \\ &= (2(a + b + c + d + e + f), -1(a + b + c + d + e + f))\end{aligned}$$

Therefore this addition is associative, and so $A2$ holds.

$A3$: Let $(a, b), (c, d) \in V$. Then

$$\begin{aligned}(a, b) + (c, d) &= (2(a + b + c + d), -1(a + b + c + d)) \\ &= (2(c + d + a + b), -1(c + d + a + b)) \\ &= (c, d) + (a, b)\end{aligned}$$

Therefore $A3$ holds.

$A4$: I claim that $A4$ fails by proving the following proposition:

Proposition. *There is no vector $(x, y) \in V$ such that $(1, 0) + (x, y) = (1, 0)$.*

Proof. If such a vector was in V, then

$$\begin{aligned}(1, 0) + (x, y) &= (1, 0) \\ (2(1 + x + y), -1(1 + x + y)) &= (1, 0) \\ (2 + 2x + 2y, -1 - x - y) &= (1, 0)\end{aligned}$$

Therefore $2 + 2x + 2y = 1$ and $-1 - x - y = 0$. Simplifying, we get the system

$$2x + 2y = -1$$

$$x + y = -1$$

which has no solution. Therefore no such vector can exist. □

Since the above proposition holds for $(1, 0)$, there is no zero vector.

$A5$: If there is no zero vector, then $A5$ can not hope to hold.

$M1$: Let $k \in \mathbb{R}$, $(x, y) \in V$. Then

$$k(x, y) = (kx, ky) \in V.$$

Therefore $M1$ holds.

$M2$: Let $k \in \mathbb{R}$, and let $(a, b), (c, d) \in V$. Then

$$\begin{aligned}
k((a, b) + (c, d)) &= k(2(a + b + c + d), -1(a + b + c + d)) \\
&= (2k(a + b + c + d), -k(a + b + c + d)) \\
&= (2(ak + bk + ck + dk), -1(ak + bk + ck + dk)) \\
&= (ak, bk) + (ck, dk) \\
&= k(a, b) + k(c, d)
\end{aligned}$$

Therefore $M2$ holds.

$M3$: I claim that $M3$ fails. To show this, I need to produce two real numbers r and s, and a pair $(x, y) \in V$ such that $(r + s)(x, y) \neq r(x, y) + s(x, y)$.

Let $r = s = 1$, and let $(x, y) = (1, 0)$. Then

$$\begin{aligned}
(r + s)(x, y) &= (1 + 1)(1, 0) \\
&= 2(1, 0) \\
&= (2, 0),
\end{aligned}$$

but...

$$\begin{aligned}
r(x, y) + s(x, y) &= 1(1, 0) + 1(1, 0) \\
&= (1, 0) + (1, 0) \\
&= (2(1 + 0 + 1 + 0), -1(1 + 0 + 1 + 0)) \\
&= (4, -2) \neq (2, 0)
\end{aligned}$$

Therefore $M3$ fails.

$M4$: Let $r, s \in \mathbb{R}$, $(x, y) \in V$. Then

$$\begin{aligned}
r(s(x, y)) &= r(sx, sy) \\
&= (r(sx), r(sy)) \\
&= ((rs)x, (rs)y) \\
&= (rs)(x, y).
\end{aligned}$$

Therefore $M4$ holds.

$M5$: For any $(x, y) \in V$,

$$1(x, y) = (1x, 1y) = (x, y).$$

Therefore $M5$ holds.

Thus, V is not a vector space. All the axioms hold except $A4$, $A5$, and $M3$.