2.2. Limits of Functions

We now start the real content of the course.

A limit is a tool for describing how (real-valued) functions behave close to a point.

We write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of $f(x)$, as x approaches a, equals L",

if the values of $f(x)$ can be made as close as we like to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.

A limit involves what is going on around a point, and does not care what happens at it.

We will talk more soon about how to calculate the limits exactly. Guessing is never the way to go in practice, but if you work through the guessing examples in the book, you may have a better idea of what is going on and how limits work.

One-sided Limits

Write

$$\lim_{x \to a^-} f(x) = L$$

and say that the **left-hand limit of** $f(x)$ **as** x **approaches** a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit of** $f(x)$ **as** x **approaches** a is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x greater than a.

So,

$$\lim_{x \to 0^-} H(x) = 0$$

and

$$\lim_{x \to 0^+} H(x) = 1.$$

For any function f and any a, the general limit $\lim_{x \to a} f(x)$ exists and equals L if and only if both the left-hand the right-hand limits exist, and

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Infinite Limits

In general, and formally, let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a but not equal to a.

Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a but not equal to a.

Similar and natural definitions can be made for one sided limits as well:

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

2.3. The Limit Laws

Suppose c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then,

- $\displaystyle\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
- $\displaystyle\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
- $\displaystyle\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
- $\displaystyle\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$
- $\displaystyle\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (if $\lim_{x\to a} g(x) \neq 0$).
- $\displaystyle\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$

Theorem. If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

This now allows us to actually find the limit of many functions.

Theorem. If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

The Squeeze Theorem. If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a), and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then

$$\lim_{x\to a} g(x) = L.$$

The idea of continuity is basic: a function is continuous if it's graph can be drawn without lifting your pencil off the paper. However we now formalize this:

A function f is said to be **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a).$$

Notice that three things are implied by this definition for a function f to be continuous at a:

1. $f(a)$ is defined, that is, a is in the domain of f
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$.

If a function is not continuous at a, we say it is **discontinuous at** a, or f has a **discontinuity** at a.

If a function is discontinuous, there are a number of special ways it can be discontinuous.

A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a),$$

and f is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a).$$

A function is **continuous on an interval** if it is continuous at every number in the interval (if f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left, as appropriate).

Theorem. If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

- $f+g$
- $f-g$
- cf
- fg
- $\frac{f}{g}$ (if $g(a) \neq 0$).

Theorem. Every polynomial is continuous everywhere, that is, on the whole real line.

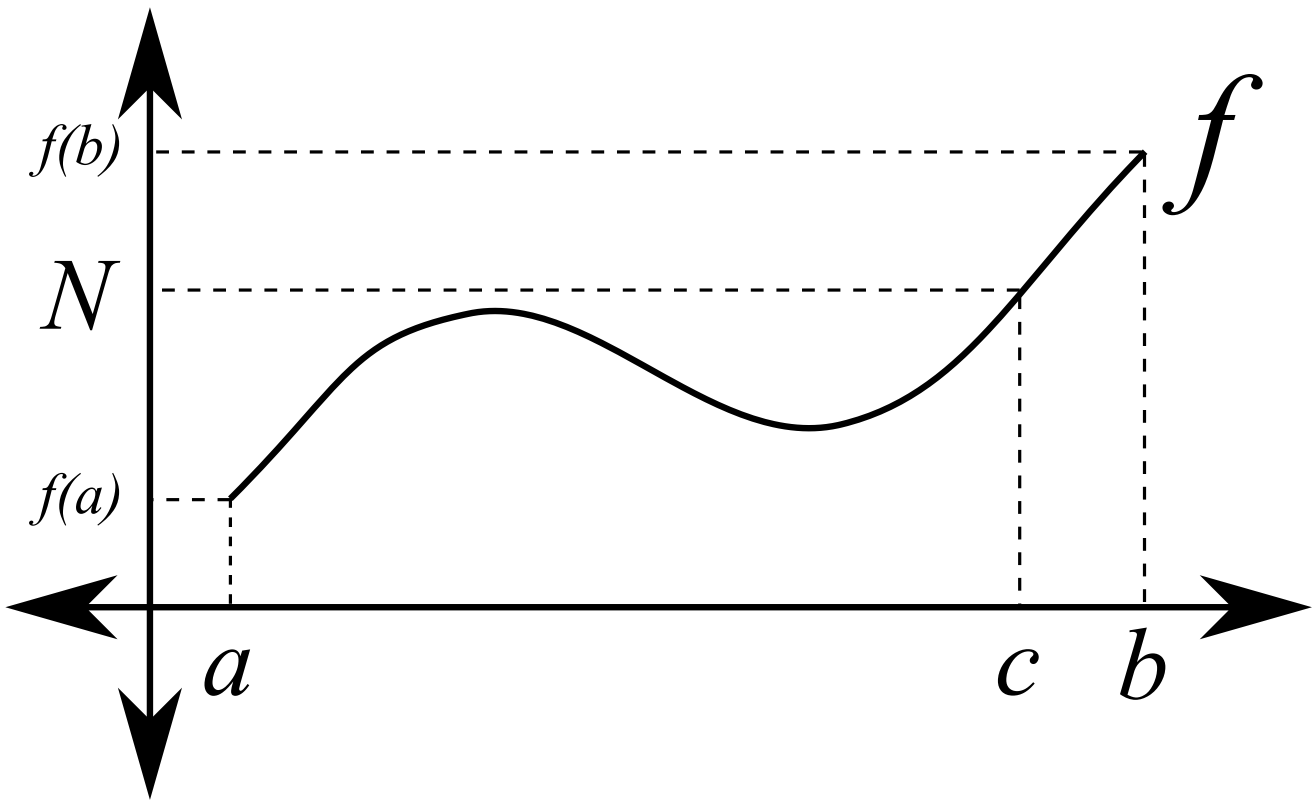
Theorem. Every rational function is continuous at every point in it's domain.

Theorem. The following types of functions are continuous at every number in their domains:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Inverse Trigonometric functions
- Exponential functions
- Logarithmic functions

Theorem. If g is continuous at a and f is continuous at $g(a)$, then the composition $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Theorem. (Intermediate Value Theorem) Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$ where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.



2.6. Limits at Infinity: Horizontal Asymptotes

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x to be sufficiently large.

Similarly,

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

Theorem. If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is any rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Consequence. For any $r \in \mathbb{Z}^+$,

$$\lim_{x\to\infty} \frac{1}{x^r} = \lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Infinite limits at infinity

The notation

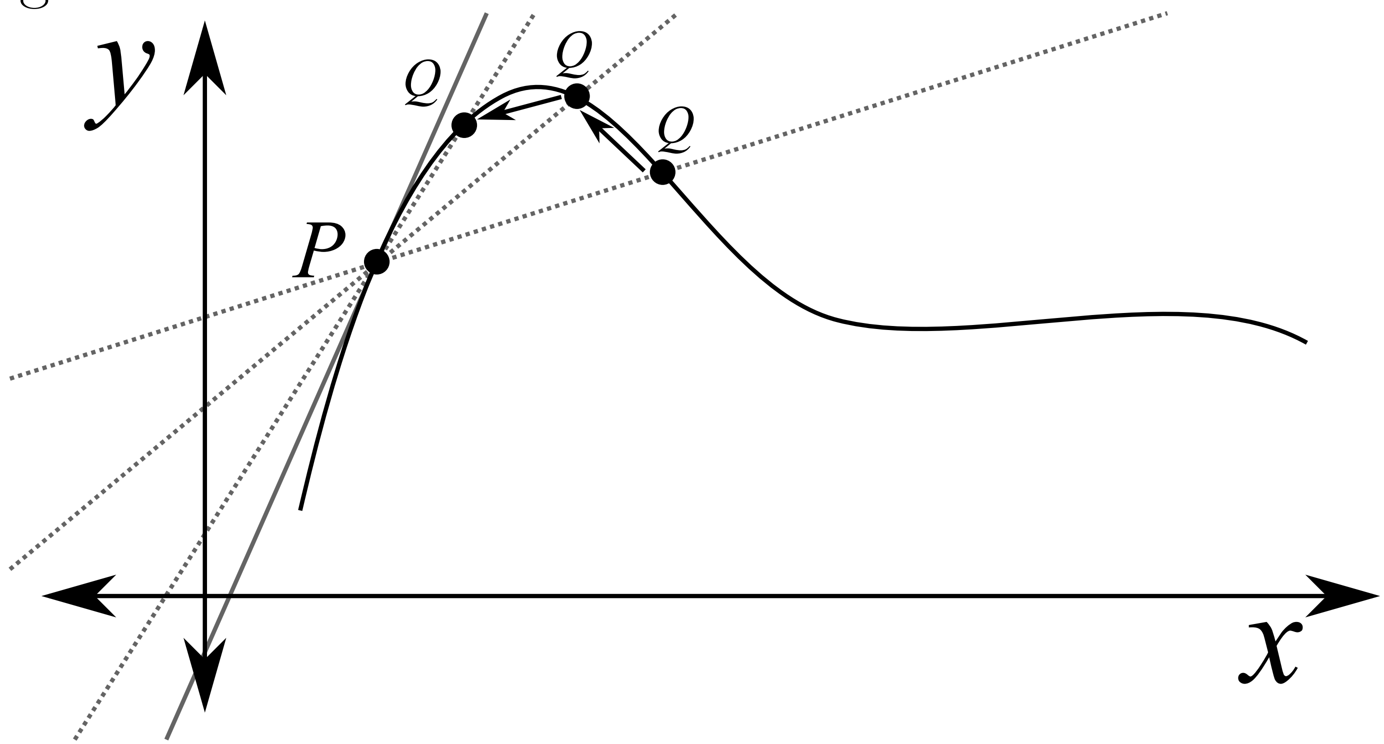
$$\lim_{x \to \infty} f(x) = \infty$$

means that the values of $f(x)$ become large as x becomes large. Similarly, we define the other notations:

$$\lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

The major purpose of limits in this course is to be able to talk about *instantaneous* rates of change.

Given any two points, one can easily determine the average rate of change between them:



Fix a function $f(x)$, and a point $P = (a, f(a))$ on the curve. Then we can find the rate of change from P to any other point $Q = (x, f(x))$ by the formula:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}.$$

Now we want to let Q approach P. The rate of change will approach closer and closer to the rate of the change of the **tangent** at P. The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

There is another version that is sometimes easier to use. Let

$$h = x - a.$$

Then

$$x = a + h,$$

so the same formula above becomes

$$\begin{aligned} m_{PQ} &= \frac{f(x) - f(a)}{x - a} \\ &= \frac{f(a+h) - f(a)}{a + h - a} \\ &= \frac{f(a+h) - f(a)}{h} \end{aligned}$$

Now, we can rewrite the slope of the tangent line as

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Derivatives.

The **derivative of a function** f **at a number** a, denoted $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

The derivative is also known as the instantaneous rate of change with respect to x.

2.8. The derivative as a function.

In the last section, we considered the derivative of a function f at a fixed number a:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Here we change our point of view and let the number a vary. If we replace a in the above equation by a variable x, we get

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For each x for which this limit exists, define the function that maps x to this number $f'(x)$. Then f' is a new function, called the **derivative of** f.

A function f is called **differentiable at** a if $f'(a)$ exists. It is differentiable on the open interval (a, b) [or (a, ∞), or $(-\infty, a)$, or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Theorem. If f is differentiable at a, then f is continuous at a.

Proof. To show that f is continuous at a, it must be shown that

$$\lim_{x \to a} f(x) = f(a).$$

Since f is differentiable at a,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. So,

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a}(f(a) + f(x) - f(a)) && \text{(add/sub } f(a)), \\
&= \lim_{x \to a} f(a) + \lim_{x \to a}(f(x) - f(a)) \\
&= f(a) + \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) && \text{(mult/div by } x - a), \\
&= f(a) + \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a}(x - a) \\
&= f(a) + f'(a) \times 0,
\end{aligned}$$

$$= f(a).$$

Therefore f is continuous at a. $\square$

Note the converse of this theorem is not true. We showed a few slides ago that $f(x) = |x|$ is not differentiable at 0, but

$$\lim_{x \to 0} |x| = 0 = f(0)$$

and therefore $|x|$ is continuous at 0.

<u>Make sure you know the direction this theorem goes.</u>

Question: How can a function look if it is not differentiable?