Applications of Differentiation

Definitions. A function f has an **absolute maximum** (or **global maximum**) at c if for all x in the domain D of f, $f(c) \geq f(x)$. The number $f(c)$ is called the **maximum value** of f on D.

Similarly, f has an **absolute minimum** (or **global minimum**) at c if for all x in D, $f(c) \leq f(x)$, and the number $f(c)$ is called the **minimum value** of f on D.

The maximum and minimum values of f are called the **extreme values** of f.

Definitions. A function f has an **local maximum** (or **relative maximum**) at c if when x is near c, (that is, x is in some open interval containing c), $f(c) \geq f(x)$.

Similarly, f has an **local minimum** (or **relative minimum**) at c if when x is near c, $f(c) \leq f(x)$.

The Extreme Value Theorem. If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's Theorem. If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Definition. A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

- If f has a local maximum or minimum at c, then c is a critical number of f.
- But a critical number might not be a local minimum or a local maximum.

<u>Finding Absolute Min's and Max's</u>

The Closed Interval Method. To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute max, and the smallest is the absolute min.

THE MEAN VALUE THEOREM.

A precursor to the Mean Value Theorem is Rolle's theorem. Rolle's theorem can be viewed as a special case of the Mean Value theorem, but it is actually used to prove the Mean Value Theorem, and so it is presented first.

Rolle's Theorem. Let f be a function that satisfies the following three hypotheses:

- f is continuous on the closed interval $[a, b]$.
- f is differentiable on the open interval (a, b).
- $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem. Let f be a function that satisfies the following hypotheses:

- $\circ$ f is continuous on the closed interval $[a, b]$.
- $\circ$ f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The Mean Value Theorem allows us to prove a number of theorems regarding how the derivative of a function affects the shape of its graph.

Theorem. If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Proof. Let x_1 and x_2 be any two numbers in (a, b) with $x_1 < x_2$. Since f is differentiable on (a, b), it must be differentiable on (x_1, x_2) and continuous on $[x_1, x_2]$. By applying the Mean Value Theorem to f on the interval $[x_1, x_2]$, we get a number c such that $x_1 < c < x_2$ and

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

But we assumed that $f'(x) = 0$ for all x in (a, b). Therefore, $f(x_2) - f(x_1) = 0$, and so $f(x_2) = f(x_1)$. Therefore, for *any* two numbers x_1, x_2 in (a, b), $f(x_1) = f(x_2)$, and so the function is constant on (a, b).

Theorem. If $f'(x) > 0$ on an interval, then f is increasing on that interval.

Proof. Let x_1 and x_2 be any two numbers in the interval, $x_1 < x_2$. According to the definition of an increasing function, we have to show that $f(x_1) < f(x_2)$.

Because we are given that $f'(x) > 0$, we know that f is differentiable on the interval (specifically on $[x_1, x_2]$, and so by the Mean Value Theorem, there is a number c, $x_1 < c < x_2$, such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) > 0$, the fraction on the right is positive. Since $x_2 > x_1$, $x_2 - x_1 \geq 0$, and so $f(x_2) - f(x_1) > 0$ too. Thus $f(x_2) > f(x_1)$, and so the function is increasing.

Theorem. If for a function f on some interval I, $f'(x) < 0$, then f is decreasing on I.

Proof. Let x_1 and x_2 be any two numbers in the interval, $x_1 < x_2$. According to the definition of a decreasing function, we have to show that $f(x_1) > f(x_2)$.

Because we are given that $f'(x) < 0$, we know that f is differentiable on the interval (and specifically on $[x_1, x_2]$, and so by the Mean Value Theorem, there is a number c, $x_1 < c < x_2$, such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since $f'(c) < 0$, the fraction on the right is negative. Since $x_2 > x_1$, $x_2 - x_1 \geq 0$, and so $f(x_2) - f(x_1) < 0$. Thus $f(x_1) > f(x_2)$, and so the function is decreasing.

Recall: A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

To look for local minimums and maximums, we have the following test:

The First Derivative Test. Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c, or negative on both sides), then f has no local maximum or minimum at c.

Definitions. If the graph f lies above all of its tangent on an interval I, then it is called **concave up** on I. If the graph of f lies below all of its tangents on I, then it is called **concave down** on I. A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there, and the curve changes from concave up to concave down, or from concave down to concave up at P.

To determine the concavity of a function at a given point, we have the following test:

Concavity Test.

(a) If for all x in some interval I, $f''(x) > 0$, then the graph of f is concave upward on I.

(b) If for all x in some interval I, $f''(x) < 0$, then the graph of f is concave down on I.

As a consequence of the concavity test, we have the following second derivative test for finding local minimums and maximums:

Second Derivative Test. Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

SUMMARY OF CURVE SKETCHING

The following list is intended as a guide for sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function, but the list should provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Find the domain.

B. Find the x and y intercepts

C. Check Symmetry (even or odd function)

D. Check for horizontal asymptotes

E. Check for vertical asymptotes

F. Check the intervals of increase and decrease

G. Find local maximum and minimum values

H. Find intervals of concavity and points of inflection

4.7. Optimization Problems

One major application of absolute extrema we have here is also referred to as an **optimization problem**. This is a problem in which we are attempting to accomplish some goal in the "best possible way" (whatever that means in context); these are in general...

WORD PROBLEMS

Some people find it hard to distinguish between Optimization problems and Related Rates problems, since both are word problems. The key is that in a Related Rates problem, quantities are changing over time. In an optimization problem, nothing is "happening over time"; the problem takes place before anything is done.

Steps in solving an optimization problem

1. Understand the problem
   - draw a picture
   - determine knowns and unknowns
   - understand question being asked (put yourself in the scenario)
   - what quantity is being maximized or minimized?
2. Create a variable for the value being maximized or minimized (lets call it Q for now), and for any other variables
3. Express Q as a function of the other variables and constants
4. Use relationships given to express Q as a function of exactly **one** other unknown
5. Find the absolute maximum or minimum (as the question asks) for this function using techniques from 4.1 - 4.3 (*e.g.*, Closed Interval Method), or...

Using the first derivative test, when there is only one critical point, we can find absolute extremes quite quickly:

The First Derivative Test for Absolute Extreme Values. Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f'(x) > 0$ for all $x < c$ and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute maximum value of f.

(b) If $f'(x) < 0$ for all $x < c$ and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute minimum value of f.

In the same way the second derivative test can be used.

The Second Derivative Test for Absolute Extreme Values. Suppose that c is a critical number of a continuous function f defined on an interval.

(a) If $f''(x) < 0$ for all x in the domain, then $f(c)$ is the absolute maximum value of f.

(b) If $f''(x) > 0$ for all x in the domain, then $f(c)$ is the absolute minimum value of f.

<u>ANTIDERIVATIVES</u>

There are a number of real life scenarios where the rate of change of something is readily known, but the value is not. In mathematical terms, the is saying that the derivative of a function is known, but the function itself is not. We now discuss “going backwards” from a derivative back to the function.

Definition. A function F is called an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem. If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

We use the symbol $\int f(x)dx$ to denote the operation "find the most general antiderivative of f(x)".

This notation is called the **indefinite integral**. The symbol $\int$ is called an **integral sign**. It is an elongated S. In the notation $\int f(x)dx$, $f(x)$ is called the **integrand**. The symbol dx has no official meaning by itself.

TABLE OF ANTIDERIVATIVES

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec^2 x$	$\tan x$
$x^n (n \neq -1)$	$\frac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\frac{1}{x}$	$\ln \lvert x \rvert$	e^x	e^x
$\cos x$	$\sin x$		

INDEFINITE INTEGRAL RULES

$\int cf(x)dx = c\int f(x)dx$ $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

$\int kdx = kx + C$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ $\int \frac{1}{x}dx = ln|x| + C$

$\int e^x dx = e^x + C$ $\int a^x dx = \frac{a^x}{\ln a} + C$

$\int \sin x dx = -\cos x + C$ $\int \cos x dx = \sin x + C$

$\int \sec^2 x dx = \tan x + C$ $\int \csc^2 x dx = -\cot x + C$

$\int \sec x \tan x dx = \sec x + C$ $\int \csc x \cot x dx = -\csc x + C$