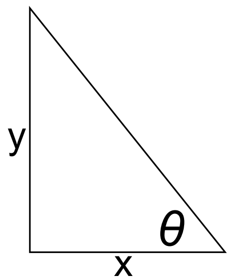
Name: ______________________________ Student Number: ____________________

Answer all questions and show all your work. No calculators allowed. (Total Marks: 26).
You have 20 minutes to complete the quiz.

1. Let θ (in radians) be an acute angle in a right traingle, and let x be the length of the side adjacent to θ (not the hypotenuse), and let y be the length of the side opposite to θ (see diagram).



[4] (a) Suppose also that x and y vary with time. How are $\frac{d\theta}{dt}$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$ related?

Solution:

$$\begin{aligned}\tan(\theta) &= \frac{y}{x}\\ \frac{d}{dt}\tan(\theta) &= \frac{d}{dt}\left(\frac{y}{x}\right)\\ \sec^2(\theta)\frac{d\theta}{dt} &= \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}.\end{aligned}$$

[6] (b) At a certain instance, $x = 2$ units and it is increasing at 1 unit / sec, while $y = 2$ units and is decreasing at $\frac{1}{4}$ units / sec. How fast is θ changing at that instant? Is θ increasing or decreasing at that moment?

Solution: At this moment, $x = 2$, $y = 2$, $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = -\frac{1}{4}$. Therefore,

$$\begin{aligned}\sec^2(\theta)\frac{d\theta}{dt} &= \frac{2(-1/4) - 2(1)}{2^2} = \frac{-5}{8}\\ \frac{d\theta}{dt} &= \frac{-5}{8\sec^2(\theta)} = \frac{-5\cos^2\theta}{8}.\end{aligned}$$

At this particular moment in time, the triangle has side lengths 2 and 2, and therefore $\theta = \pi/4$. Therefore, at this moment, θ is changing at a rate of

$$\frac{-5\cos^2(\pi/4)}{8} = \frac{-5\left(\frac{1}{2}\right)}{8} = \frac{-5}{16} \text{ rads/sec.}$$

In other words, θ is decreasing by $\frac{5}{16}$ rads/sec.

2. Let $f(x) = (\ln(x^2))^2$.

[4] (a) Find all real numbers a such that $f(a) = 1$.

Solution:

$$f(a) = (\ln(a^2))^2 = 1$$
$$\ln(a^2) = \pm 1$$

$$\ln(a^2) = 1 \implies a^2 = e \implies a = \pm\sqrt{e}$$
$$\ln(a^2) = -1 \implies a^2 = e^{-1} \implies a = \pm\sqrt{1/e}$$

[4] (b) Find $f'(x)$. Do not simplify your answer.

Solution:

$$f'(x) = 2(\ln(x^2))\frac{1}{x^2}(2x) = \frac{4\ln(x^2)}{x}$$

[8] 3. Show that the line $y = 2x - 1$ is tangent to a circle centered at $(8, 0)$, and find the equation of this circle. Reminder: The formula for a circle centered at (a, b) with radius r is $(x-a)^2 + (y-b)^2 = r^2$.

Solution:

A circle centered at $(8, 0)$ has the form

$$(x - 8)^2 + y^2 = r^2.$$

Differentiating, we have that

$$2(x - 8) + 2yy' = 0.$$

Solving for y', we get $y' = \frac{-2(x-8)}{2y} = \frac{8-x}{y}$.

We want $y' = 2$, so we solve:

$$2 = \frac{8 - x}{y}$$
$$2y = 8 - x$$
$$y = 4 - \frac{1}{2}x.$$

Also at this point, $y = 2x - 1$, so we have

$$2x - 1 = 4 - \frac{1}{2}x$$
$$2x + \frac{1}{2}x = 4 + 1$$
$$\frac{5}{2}x = 5$$
$$x = 2.$$
$$y = 2(2) - 1$$
$$= 3.$$

So the point $(2, 3)$ is on the circle. The radius is then $\sqrt{(8-2)^2 + (0-3)^2} = \sqrt{6^2 + 3^2} = \sqrt{45}$, and so the circle's formula is

$$(x - 8)^2 + y^2 = 45.$$