Extrema Problems

1. Divide 20 into two parts (not necessarily integers) such that the product of one part with the square of the other shall be a ~~minimum.~~ maximum

2. Find the number which most exceeds its square.

3. An open rectangular box is to be made from a piece of cardboard 8 in. wide and 15 in. long by cutting a square from each corner and bending up the sides. Find the dimensions of the box of largest volume.

4. A poster is to contain 50 in^2 of printed matter with margins of 4 in. each at top and bottom and 2 in. at each side. Find the overall dimensions if the total area of the poster is to be a minimum.

5. An oil can is to be made in the form of a right circular cylinder to contain 16π in^3. What dimensions of the can require the least amount of material?

6. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the curve $y = 12 - x^2$.

7. A box with square base and open top is to hold 32 in^3. Find the dimensions that require the least amount of material.

8. Find the height and radius of the right circular cylinder of maximum volume which can be inscribed in a sphere of radius $\sqrt{3}$ ft.

9. A right circular cone has altitude 12 ft and radius of base 6 ft. A cone is inscribed with its vertex at the center of the base of the given cone and its base parallel to the base of the given cone. Find the dimensions of the cone of maximum volume that can be so inscribed.

10. An aquarium is to be made 6 ft high and is to have a volume of 750 cu. ft. The base, ends, and back are to be made of slate, but the front is to be made of plate glass, which costs 3/2 times as much as slate per sq. ft. What dimensions should be chosen to make the cost of raw materials a minimum?

11. A playing field is to be built in the shape of a rectangle plus a semicircular area at each end. A 440 yd race track is to form the perimeter of the field. Find the dimensions of the field if the rectangular part is to have as large an area as possible.

12. The strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius r.

13. A motorist is stranded in a desert 5 mi from a point A, which is the point on a long straight road nearest to him. He wishes to get to a point B on the road. If he can travel at 15 mi/hr on the desert and 39 mi/hr on the road, find the point at which he must meet the road to get to B in the shortest possible time if

    (a) B is 5 mi from A,
    (b) B is 10 mi from A,
    (c) B is 1 mi from A.

14. At midnight, ship B was 90 mi due south of ship A. Ship A sailed east at 15 mi/hr and ship B sailed north at 20 mi/hr. At what time were they closest together?

15. A farmer wishes to construct 9 pig pens by fencing a rectangular region and then subdividing the region by 8 fences parallel to one of the sides. If the farmer has 400 m of fencing, what dimensions of the region will give the largest total area?

16. A power company wishes to lay a cable from point A on one side of a river (100 m wide) to point B on the other side which is 500 m downstream from A. If underwater cable costs \$10 per meter and land cable costs \$5 per meter, how should the cable be laid in order to minimize the cost?

17. A real estate company owns 200 apartments which are fully occupied when the rent is \$300 per month. The company estimates that for each \$2 increase in rent, one less apartment will be rented. Each occupied apartment requires \$20 of maintenance per month. What rent should be charged to obtain the largest net income?

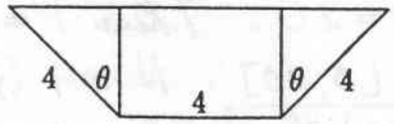


Figure 1: for Problem 21

18. A package can be sent by parcel post only if the sum of its length and girth (the perimeter of the base) is not more than 96 in. Find the dimensions of the box of maximum volume that can be sent if the base of the box is a square.

19. The cost of fuel in running a locomotive is proportional to the square of the speed and is \$25 per hour for a speed of 25 mi/hr. Other costs amount to \$100 per hour, regardless of the speed. Find the speed which will make the cost per mile a minimum.

20. A wall 8 ft high is 27/8 ft from a house. Find the shortest ladder which will reach from the ground to the house when leaning over the wall.

21. A gutter is to be made out of a long sheet of metal 12 in. wide by turning up strips of width 4 in. along each side so that they make equal angles θ with the vertical. For what value of θ will the carrying capacity be greatest? (See Figure 1.)

(omit) 22. A tablet 7 ft high is placed on a wall with its base 9 ft above the level of an observer's eye. How far from the wall should an observer stand in order that the angle subtended at his eye by the tablet be a maximum?

Solutions [Global means Absolute]

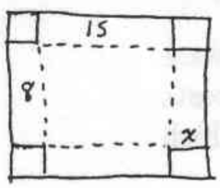
1. Let x and y be the two numbers and P the required product. Then $P = xy^2$ and $x+y=20$. Thus $P = f(y) = (20-y)y^2$ $= 20y^2 - y^3$ and $\operatorname{dom} f = [0,20]$. Now $f'(y) = 40y - 3y^2$ $= y(40-3y)$ and

y	0	20	$\frac{40}{3}$
$f(y)$	0	0	$\frac{32000}{27}$

. Thus the two parts are $\frac{40}{3}$ and $\frac{20}{3}$.

2. Let x be the number and let E be the excess. Then $E = f(x) = x - x^2$, $\operatorname{dom} f = \mathbb{R}$. Now $f'(x) = 1-2x$

$+ \quad \frac{1}{2} \quad - \quad f'$ so f has global max at $\frac{1}{2}$. Thus number is $\frac{1}{2}$

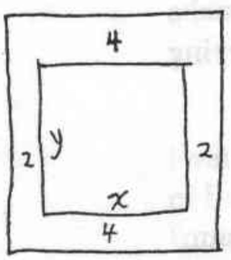
3. 

Let x be side of square and V the volume. Then $V = f(x) = (8-2x)(15-2x)x$, $\operatorname{dom} f = [0,4]$. Now $f(x) = 120x - 46x^2 + 4x^3$ so $f'(x) = 12x^2 - 92x + 120$ $= 4(3x-5)(x-6)$.

x	0	4	$\frac{5}{3}$
$f(x)$	0	0	$\frac{2450}{27}$

Thus f has global max at $\frac{5}{3}$.

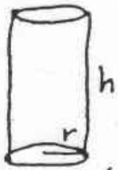
Thus dimensions are $\frac{14}{3} \times \frac{35}{3} \times \frac{5}{3}$.

4. 

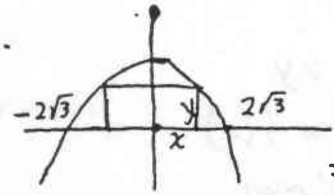
Let A be area of poster. Then $A = (x+4)(y+8)$ and $xy = 50$. Thus $A = f(x) = (x+4)(\frac{50}{x}+8)$ $= 82 + 8x + \frac{200}{x}$ and $\operatorname{dom} f = (0,\infty)$. Now $f'(x) = 8 - \frac{200}{x^2} = \frac{8(x-5)(x+5)}{x^2}$

$0 \quad - \quad 5 \quad +$ so f has global min at 5.

Thus dimensions should be 9×18

5. 

Let A be surface area of can, r radius of base and h height. Then $A = 2\pi r h + 2\pi r^2$ and $\pi r^2 h = 16\pi$. Thus $A = f(r) = 2\pi r(\frac{16}{r^2}) + 2\pi r^2$ $= 2\pi(r^2 + \frac{16}{r})$ and $\operatorname{dom} f = (0,\infty)$. Now $f'(r) = 2\pi(2r - \frac{16}{r^2})$ $= 4\pi(\frac{r^3-8}{r^2}) = 4\pi\frac{(r-2)(r^2+2r+4)}{r^2}$ $\quad 0 \quad - \quad 2 \quad + \quad$ so f has global min at 2. Thus can should have dimensions radius 2 in and height 4 in.

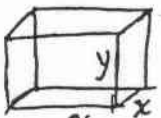
6. 

Let A be area of rectangle. Then $A = 2xy$ and $y = 12 - x^2$. Thus $A = f(x) = 2x(12 - x^2)$ $= 24x - 2x^3$ and dom $f = [0, 2\sqrt{3}]$.

Now $f'(x) = 24 - 6x^2 = 6(2-x)(2+x)$

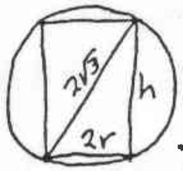
x	0	$2\sqrt{3}$	2
$f(x)$	0	0	32

Thus f has global max at 2. The area is 32.

7. 

Let M be amount of material. Then $M = 4xy + x^2$ and $x^2y = 32$. Thus $M = f(x) = 4x\left(\frac{32}{x^2}\right) + x^2$ $= \frac{128}{x} + x^2$ and dom $f = (0, \infty)$. Now $f'(x) = 2x - \frac{128}{x^2} = \frac{2(x-4)(x^2+4x+16)}{x^2}$

$\underline{\quad - \;\; 4 \;\; + \quad}$ so f has global min at 4. Thus box should have dimensions $4 \times 4 \times 2$ in.

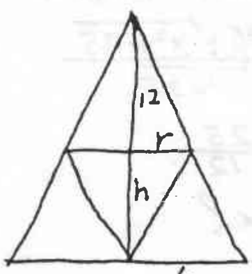
8. 

Let h be height of cylinder and r the radius and V the volume. Then $V = \pi r^2 h$ and $h^2 + (2r)^2 = (2\sqrt{3})^2$

Thus $V = f(h) = \pi h\left(\frac{12-h^2}{4}\right) = \frac{\pi}{4}(12h - h^3)$ and dom $f = [0, 2\sqrt{3}]$. Now $f'(h) = \frac{\pi}{4}(12 - 3h^2) = \frac{3\pi}{4}(2-h)(2+h)$

h	0	$2\sqrt{3}$	2
$f(h)$	0	0	4π

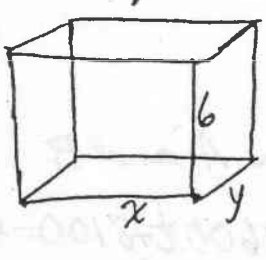
so f has global max at 2. Thus cylinder has height 2 ft and radius $\sqrt{2}$ ft.

9. 

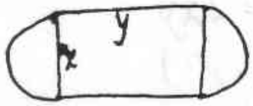
Let r be radius of cone and h the height and V the volume. Then $V = \frac{1}{3}\pi r^2 h$ and $\frac{r}{6} = \frac{12-h}{12}$ or $h = 12 - 2r$. Thus $V = f(r)$ $= \frac{1}{3}\pi r^2(12 - 2r) = \frac{\pi}{3}(12r^2 - 2r^3)$ and dom $f = [0, \infty)$

Now $f'(r) = \frac{\pi}{3}(24r - 6r^2) = 2\pi r(4 - r)$ $\quad \underline{0 \;\; + \;\; 4 \;\; -}$

so f has global max at 4. Thus cone has radius 4 ft. and height 4 ft.

10. 

Let C be the cost. Then $C = 2(6y) + 6x + \frac{3}{2}(6x) + xy$ and $6xy = 750$. Thus $C = f(x) = 15x + 750 + 12\left(\frac{125}{x}\right)$ and dom $f = (0, \infty)$. Now $f'(x) = 15 - \frac{12(125)}{x^2}$ $= \frac{15(x-10)(x+10)}{x^2}$ $\quad \underline{0 \;\; - \;\; 10 \;\; +}$ so f has global min at 10. Thus dimensions are $6 \times 10 \times 12\frac{1}{2}$ ft.

11. 

Let A be area of rectangle. Then $A = xy$ and $2y + 2\pi\left(\frac{x}{2}\right) = 440$. Thus $A = f(x)$ $= x\left(220 - \frac{\pi}{2}x\right) = 220x - \frac{\pi}{2}x^2$ and dom $f = \left[0, \frac{440}{\pi}\right]$. Now $f'(x) = 220 - \pi x$

x	0	$\frac{440}{\pi}$	$\frac{220}{\pi}$
$f(x)$	0	0	$110\left(\frac{220}{\pi}\right)$

Thus f has a global max at $\frac{220}{\pi}$

Therefore dimensions are $\frac{220}{\pi} \times 110$ yd.

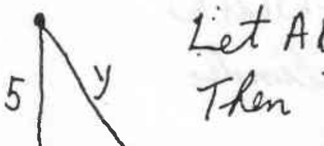
12. 

Let x be the width of log and y its depth. and S its strength. Then $S = kxy^2$ (where k some real constant) and $x^2 + y^2 = (2r)^2$

Thus $S = f(x) = kx(4r^2 - x^2) = k(4r^2x - x^3)$ and dom $f = [0, 2r]$. Now $f'(x) = k(4r^2 - 3x^2)$

x	0	$2r$	$\frac{2r}{\sqrt{3}}$
$f(x)$	0	0	$\frac{16kr^3}{3\sqrt{3}}$

so f has global max at $\frac{2r}{\sqrt{3}}$. Thus log has width $\frac{2r}{\sqrt{3}}$ and depth $\frac{2\sqrt{2}\,r}{\sqrt{3}}$.

13. 

Let AB have length l and let total time be T. Then $T = \frac{y}{15} + \frac{l - x}{39}$ and $y^2 = x^2 + 25$. Thus

$$T = f(x) = \frac{\sqrt{x^2+25}}{15} + \frac{l-x}{39} \text{ and dom } f = [0, l]$$

Now $f'(x) = \frac{x}{15\sqrt{x^2+25}} - \frac{1}{39} = \frac{39x - 15\sqrt{x^2+25}}{(15)(39)\sqrt{x^2+25}}$

and $f'(x) = 0$ iff $39x - 15\sqrt{x^2+25} = 0$ or $x = \frac{25}{12}$.

(a) dom $f = [0, 5]$

x	0	5	$\frac{25}{12}$
$f(x)$	$\frac{18}{39}$	$\frac{\sqrt{2}}{3}$	$\frac{17}{39}$

Thus f has global min at $\frac{25}{12}$

(b) dom $f = [0, 10]$

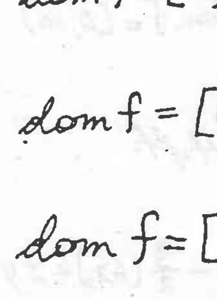
x	0	10	$\frac{25}{12}$
$f(x)$	$\frac{23}{39}$	$\frac{\sqrt{5}}{3}$	$\frac{22}{39}$

Thus f has global min at $\frac{25}{12}$

(c) dom $f = [0, 1]$

x	0	1
$f(x)$	$\frac{14}{39}$	$\frac{\sqrt{26}}{15}$

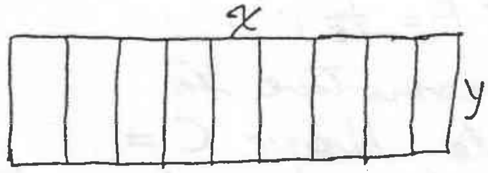
Thus f has global min at 1

14. 

Let D be square of distance between A and B. Then $D = (15t)^2 + (90 - 20t)^2 = 625t^2 - 3600t + 8100 = f(t)$

Now $f'(t) = 1350t - 3600$ $\quad$ $\underset{0}{\bullet}$ $-$ $\underset{\frac{8}{3}}{\bullet}$ $+$ $\quad$ dom $f = [0, \infty)$

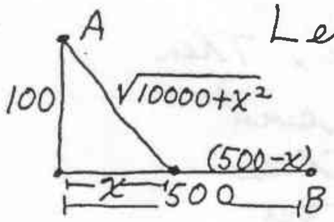
Thus f has global min at $t = \frac{8}{3}$. Thus ships closest at 2:40 A.M.

15. 

Let A be total area. Then $A = xy$ and $2x + 10y = 400$

Thus $A = f(x) = x\left(40 - \frac{x}{5}\right) = 40x - \frac{x^2}{5}$ and $\operatorname{dom} f = (0, \infty)$

Now $f'(x) = 40 - \frac{2x}{5} = \frac{2}{5}(100 - x)$ $\quad 0 \ + \ 100 \ -$ so f has global max at 100. Thus dimensions are 100×20.

16. 

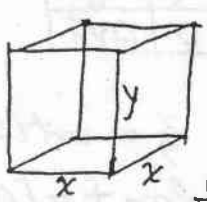
Let C be cost of cable. Then $C = f(x) = 10(\sqrt{10000+x^2}) + 5(500 - x)$ and $\operatorname{dom} f = [0, 500]$. Now $f'(x) = \frac{10x}{\sqrt{10000+x^2}} - 5 = \frac{10x - 5\sqrt{10000+x^2}}{\sqrt{10000+x^2}}$ and

$f'(x) = 0$ iff $10x - 5\sqrt{10000+x^2} = 0$ or $x = \frac{100}{\sqrt{3}}$

x	0	500	$\frac{100}{\sqrt{3}}$
$f(x)$	3500	$1000\sqrt{26}$	$2500 - \frac{300}{\sqrt{3}}$

Thus f has global min at $\frac{100}{\sqrt{3}}$

17. Let N be net income and x be the number of apartments <u>not</u> rented. The rent per apartment is $300 + 2x$ and the number of apartments rented is $200 - x$. Thus
$N = f(x) = (300 + 2x)(200 - x) - 20(200 - x) = -2x^2 + 120x + 56000$, $\operatorname{dom} f = [0, 200]$. Now $f'(x) = -4x + 120 = -4(x - 30)$ $\quad 0 \ + \ 30 \ - \ 200$ so f has global max at 30. Thus rent charged should be \$360.

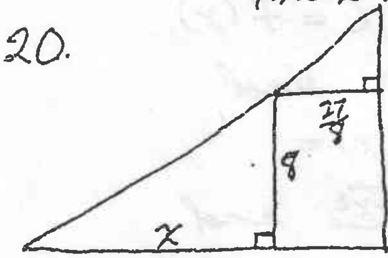
18. 

Let V be volume. Then $V = x^2 y$ and $4x + y = 96$. Thus $V = f(x) = x^2(96 - 4x) = 96x^2 - 4x^3$ and $\operatorname{dom} f = [0, \infty)$. Now $f'(x) = 192x - 12x^2 = 12x(16 - x)$ $\quad 0 \ + \ 16 \ -$

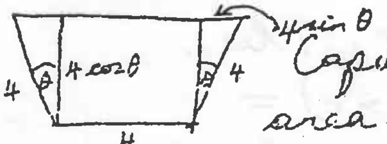
so f has global max at 16. Thus dimensions are $16 \times 16 \times 32$

19. Let F be the cost of fuel per hour. Then $F = kv^2$ and $25 = k(25)^2$ so $k = \frac{1}{25}$ and $F = \frac{1}{25}v^2$.
Thus cost per hour of running locomotive is $\frac{1}{25}v^2 + 100$. Let C be cost per mile. Now $C = \frac{\text{cost per hour}}{\text{miles per hour}}$. Thus $C = \frac{\frac{1}{25}v^2 + 100}{v} = \frac{1}{25}v + \frac{100}{v} = f(v)$ and dom $f = (0, \infty)$. Now $f'(v) = \frac{1}{25} - \frac{100}{v^2} = \frac{(v-50)(v+50)}{25v^2}$

$\underset{0}{\bullet}\ \ - \ \ \underset{50}{\bullet}\ \ +$ so f has global min at 50.
Thus speed should be 50 m.p.h.

20. 

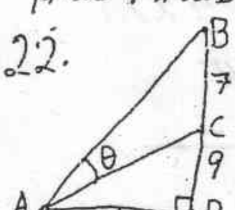
Let L be length of ladder. Then $L = \sqrt{x^2+64} + \sqrt{y^2 + \left(\frac{27}{8}\right)^2}$ and $\frac{y}{8} = \frac{\frac{27}{8}}{x}$ or $y = \frac{27}{x}$. Thus $L = f(x) = \sqrt{x^2+64} + \sqrt{\frac{27^2}{x^2} + \frac{27^2}{8^2}}$
$= \sqrt{x^2+64}\left(1 + \frac{27}{8x}\right)$ and dom $f = (0, \infty)$. Now
$f'(x) = \frac{x}{\sqrt{x^2+64}}\left(1 + \frac{27}{8x}\right) + \sqrt{x^2+64}\left(-\frac{27}{8x^2}\right) = \frac{x^3 - 216}{x^2\sqrt{x^2+64}}$
$= \frac{(x-6)(x^2+6x+36)}{x^2\sqrt{x^2+64}}$ $\ \ - \ \underset{6}{\bullet}\ +$ Thus f has global min of $\frac{125}{8}$ at $x = 6$.
Thus ladder is $\frac{125}{8}$ ft. long.

21. 

Capacity will be greatest when the cross-sectional area is greatest. Let A be the area.
Then $A = f(\theta) = 4(4\cos\theta) + 2\left[\frac{1}{2}(4\cos\theta)(4\sin\theta)\right]$
$= 16(\cos\theta + \sin\theta\cos\theta)$ and dom $f = [0, \frac{\pi}{2}]$. Then $f'(\theta) =$
$16(-\sin\theta + \cos^2\theta - \sin^2\theta) = -16(2\sin\theta - 1)(\sin\theta + 1)$. Then $f'(\theta) = 0$ when $\sin\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{6}$ and $\sin\theta = -1$ or $\theta = \frac{3\pi}{2}$.

θ	0	$\frac{\pi}{2}$	$\frac{\pi}{6}$
$f(\theta)$	16	0	$12\sqrt{3}$

Thus f has global max when $\theta = \frac{\pi}{6}$ (30°)

it) 22. 

Now $\theta = \angle BAD - \angle CAD$ and $\frac{x}{16} = \cot(\angle BAD)$, $\frac{x}{9} = \cot(\angle CAD)$
Thus $\theta = f(x) = \operatorname{arccot}\left(\frac{x}{16}\right) - \operatorname{arccot}\left(\frac{x}{9}\right)$, dom $f = [0, \infty)$
Then $f'(x) = \frac{-1}{1 + \left(\frac{x}{16}\right)^2}\left(\frac{1}{16}\right) - \frac{-1}{1 + \left(\frac{x}{9}\right)^2}\left(\frac{1}{9}\right) = \frac{-16}{256 + x^2} + \frac{9}{81 + x^2}$
$= \frac{1008 - 7x^2}{(256 + x^2)(81 + x^2)} = \frac{7(12 - x)(12 + x)}{(256 + x^2)(81 + x^2)}$ $\ \underset{0}{\bullet}\ + \ \underset{12}{\bullet}\ -$. Thus f has a global max. when $x = 12$.