Introduction to the Probabilistic Method

Robert D. Borgersen
umborger@cc.umanitoba.ca

September 23, 2005

Abstract

The Probabilistic Method is a powerful tool in discrete mathematics, combinatorics, and graph theory. Since it has become more popular, it has also been applied to problems in number theory, combinatorial geometry, linear algebra and real analysis. Pioneered by Paul Erdös, the method allows one to prove existence of a structure with particular properties by defining an appropriate probability space of structures and showing that the desired properties hold in the space with positive probability.

In this talk I will present some relevant probability theory, the probabilistic method, and some applications.

Outline

- A Ramsey Bound–Counting Proof
- A Ramsey Bound–Probabilistic Proof
- Basic Probability
- Models of Random Graphs
- The Method's Power
- Examples
- The Second Moment Method
- Example
- Threshold Functions (Time Permitting)

Ramsey Theory

Definitions: Graph, Vertex, Edge, Complete Graph K_n, Edge Coloring

The Ramsey Number $R(k, k)$ is then the smallest integer n such that however we color the edges of a K_n with two colors (say red and blue), there will always be a monochromatic subgraph K_k.

Bounding $R(k,k)$ by Counting

We will use both a counting argument, and the probabilistic method to prove the following theorem:

Theorem 1 *If* $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ *then* $R(k,k) > n$*. In fact, for* $k \geq 3$*,* $R(k,k) > 2^{\frac{k}{2}}$*.*

Proof: Counting Argument

Bounding $R(k, k)$ with the Probabilistic Method

Proof of Theorem 1: Probabilistic Argument

The two arguments are of about the same complexity in this example, but when problems become more complex, counting becomes exceedingly so, while the probabilistic method remains relatively straight forward.

Basic Probability

Definition 2 *A* **Sample Space** *(for our purposes) is a pair* (Ω, P) *where* Ω *is a finite set, and* $P : \Omega \rightarrow [0, 1]$ *satisfies*

$$\sum_{G \in \Omega} P(G) = 1$$

Definition 3 *A* **Random Variable** *is a function* $X : \Omega \rightarrow \mathbb{R}$*.*
If $X : \Omega \rightarrow \{0, 1\}$*,* X *is called an* **Indicator Random Variable***.*
Note: Often X *is written, however,* $X(G)$ *is meant.*

Definition 4 *The* **Expected Value of X** *is defined as*

$$E(X) = \sum_{G \in \Omega} X(G) \cdot P(G)$$

Theorem 5 Linearity of Expectation: *Expectation as above is a* **Linear Function**. *I.e. for any two random variables X and Y, and any $k \in \mathbb{R}$,*

$$E(kX + Y) = kE(X) + E(Y)$$

Definition 6 *The* **Variance** *of X is defined as $E((X - E(X))^2)$*

Definition 7 The ith moment of a random variable X *is defined to be $E(X^i)$. Specifically, the 1st moment of X is $E(X)$ and the 2nd moment of X is $E(X^2)$.*

Models of Random Graphs

Model A - The Binomial Model

Definition 8 *Fix $n \in \mathbb{Z}^+$, $0 \leq p \leq 1$. Let Ω be the set of all labelled graphs on n vertices. For any graph $G \in \Omega$ with q edges, define $P : \Omega \to [0,1]$ as*

$$P(G) = p^q(1-p)^{\binom{n}{2}-q}$$

Note: the binomial theorem proves immediately that (Ω, P) is a valid sample space ($P(\Omega) = 1$)

Think of this as picking edges consecutively by flipping a coin $\binom{n}{2}$ times that will land heads with probability p.

Model B - The Uniform Model

Definition 9 *Let Ω be the set of all labelled graphs with q edges, and let*

$$P(G) = \binom{\binom{n}{2}}{q}^{-1}$$

Note: when $q \sim p\binom{n}{2}$, these models are "nearly equivalent"

Model B was used in 1939 to show that graphs on n vertices with $q \sim \frac{1}{2}n \ln n$ edges are "almost surely" connected. In other words,

$$P(G \text{ is connected}) \to 1 \text{ as } n \to \infty$$

The Method's Power

The power of the method comes from it's ability to prove existence without the necessity of construction. When objects (like Graphs, or other structures) become very large, constructing them becomes an unreasonable task to undertake for man, computer, or even group of computers! At this point, we can only hope to prove existence (or non-existence). This is especially true in the studies of asymptotic behavior, like Extremal Graph Theory.

Example: Graph Theory

Theorem 10 *Let $G = (V, E)$ be a graph with n vertices and e edges. Then G contains a bipartite subgraph with at least $e/2$ edges.*

Proof:

This theorem was also extended by Noga Alon (1996) to the following tight theorem:

Theorem 11 *Let $G = (V, E)$ be a graph with n vertices and e edges. Then G contains a bipartite subgraph with at least $\frac{e}{2} + \sqrt{\frac{e}{8}} + ce^{\frac{1}{4}}$ edges.*

Example: Combinatorial Number Theory

Definition 12 *A subset* A *of an abelian group* G *is called* sum-free *if there are no* $a_1, a_2, a_3 \in A$ *such that* $a_1 + a_2 = a_3$.

Theorem 13 *(Erdös, 1965) Every set* $B = \{b_1, ..., b_n\}$ *of* n *non-zero integers contains a sum-free subset* A *with* $|A| > \frac{n}{3}$

Proof:

The Second Moment Method

Let X be a random variable ($X(G) \geq 0$). Lets say we have come across a situation involving X where we know that

$$E(X) \to 0 \quad \text{as} \quad n \to \infty$$

Can we then say that $P(X = 0) \to 1$?

Answer: YES

Lets say we have another situation where

$$E(X) \to \infty \quad \text{as} \quad n \to \infty$$

Can we then say that $P(X = 0) \to 0$?

Answer: NO! No matter how big $E(X)$ gets, it could be the case that for half the G's, $X(G) = 2E(X)$, and for the other half, $X(G) = 0$. In this case, $P(X = 0) = \frac{1}{2} > 0$!

We need a little bit more information here to be able to make the conclusion we want: we need to know about the distribution of the values. Specifically, we need to look at the variance.

Theorem 14 *(Chebyshev's Inequality) For* $t \geq 0$*,*

$$P(|X - E(X)| \geq t) \leq \frac{V(X)}{t^2}$$

Using Chebyshev's Inequality with $t = E(X)$ yields:

$$\begin{aligned} P(X = 0) &\leq P(|X - E(X)| \geq E(X)) \\ &\leq \frac{V(X)}{E(X)^2} \end{aligned}$$

So if we have that $E(X) \to 0$, and

$$lim_{n\to\infty} \frac{V(X)}{E(X)^2} = 0$$

then it must be that

$$P(X = 0) \to 0$$

Using this fact is called the *Second Moment Method*.

Second Moment Method & Threshold Functions Example

Theorem 15 *Let G be a graph with n vertices and approximately* $c\frac{1}{2}nlogn$ *edges. Then*

$$\textit{if } c > 1 \textit{ then } P(G \textit{ has isolated vertices}) \to 0$$

and

$$\textit{if } c < 1 \textit{ then } P(G \textit{ has isolated vertices}) \to 1$$

Proof:

Threshold Functions

Let $\mathcal{A}$ be the set of graphs with n vertices and q edges with property Q in the uniform model $G(n,q)$. Let c be a constant. A function $t(c,n)$ is called a *threshold function for* Q if there exists c_0 such that when $q \sim t(c,n)$

1. $c > c_0 \implies P(\mathcal{A}) \to 1$ (almost all graphs in G(n,q) have property Q), and

2. $c < c_0 \implies P(\mathcal{A}) \to 0$ (almost none of the graphs in G(n,q) have property Q)

References

- N. Alon: Bipartite Subgraphs, *Combinatorica*, **16** (1996), pg 301-311

- N. Alon and J. H. Spencer (1992): *The Probabilistic Method*. John Wiley & Sons, Inc., New York, NY. ISBN 0471535885

- E. Palmer (1985): *Graphical Evolution*. John Wiley & Sons, Inc., New York, NY. ISBN 0-471-81577-2.

- J. H. Spencer (1987): *Ten Lectures on the Probabilistic Method*. Thomson-Shore, Dexter, MI. ISBN 0-89871-213-0.

- M. S. Wackerly (1981): *Mathematical Statistics with Applications*. PWS Publishers, Boston, MA. ISBN 0-87150-410-3.