Graph Reconstruction

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Abstract

The concept of a graph is one of the most basic and readily understood mathematical concepts, and the Reconstruction Conjecture is one of the most engaging problems under the domain of Graph Theory. The conjecture proposes that every graph with at least three vertices can be uniquely reconstructed given the multiset of subgraphs produced by deleting each vertex of the original graph one by one. This conjecture has been proven true for several infinite classes of graphs, but the general case remains unsolved. In this talk, I will outline the problem, and introduce some of the most well known results.
Graph Theory

The study of Graphs ...
Graph Theory

The study of Graphs ... so what’s a graph?
$V(G) = \text{the set of vertices of } G$. 
Graph Theory

- $V(G) = \text{the set of vertices of } G$.
- $E(G) = \text{the set of edges of } G$. 
Definitions

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- $K_n$ denotes the **complete** graph on $n$ vertices (every possible edge).

- A **card** of $G$ is an unlabelled graph formed by deleting 1 vertex and all edges attached to it.

- The **deck** of $G$, $\mathcal{D}(G)$, is the collection of all $G$’s cards. Note this is in general a multiset.
Example
Example
Example
Example

![Graph Diagram]

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Graduate Seminar: Graph Reconstruction – p. 5/24
Example
Example
Example
Example

A

B

C

D

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A graph is **reconstructible** if, given it’s deck, we can uniquely determine what the original graph was.
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Equivalently, $G$ is not reconstructible if and only if there is an $H$ such that $\mathcal{D}(G) = \mathcal{D}(H)$.
Example

This graph is reconstructible.
Example

This graph is reconstructible.

The last three cards are also in the deck of
Example

This graph is reconstructible.

The last three cards are also in the deck of (these can be checked by exhaustion).
The Reconstruction Conjecture

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  Every graph with at least 3 vertices is reconstructible.
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- **We need at least three vertices:**

```
  .  .  .
     |
  .  .  .
```
The Reconstruction Conjecture

**Reconstruction Conjecture:**
Every graph with at least 3 vertices is reconstructible.

- We need at least three vertices:

  ![Graph Diagram](image)

- Note: the reconstruction conjecture is false if and only if there exist two graphs with the same deck.
Reconstructible Properties

Some properties of $G$ are easily determined from $D(G)$:
Reconstructible Properties

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- $|V(G)| = \# \text{ of cards in } \mathcal{D}(G)$.

- $|E(G)|$: Each edge in $G$ is missing from exactly two cards. Thus each edge is in exactly $n - 2$ cards, and we have

$$|E(G)| = \frac{1}{n - 2} \sum_{v \in V(G)} |E(G - v)|.$$
Further...

Lemma: $\forall v \in V(G),$

$$deg(v) = |E(G)| - |E(G - v)|.$$
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Since $|E(G)|$ can be determined from $\mathcal{D}(G)$, we can also determine the degree of $v$ for all $v \in V(G)$. 
Further...

**Lemma:** \( \forall v \in V(G), \)

\[ \text{deg}(v) = |E(G)| - |E(G - v)|. \]

- Since \( |E(G)| \) can be determined from \( \mathcal{D}(G) \), we can also determine the degree of \( v \) for all \( v \in V(G) \).

- \( \implies \) We can determine the whole degree sequence of \( G \).
Reconstruction Example

Reconstruct, if possible, one or more graphs from the following deck:
Reconstruction Example

\[ |V(G)| = \# \text{ of cards}. \]
Reconstruction Example

\[ |V(G)| = 5. \]
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\[ |E(G)| = \frac{1}{|V(G)| - 2} \sum_{v \in V(G)} |E(G - v)|. \]
Reconstruction Example

\begin{align*}
|V(G)| &= 5. \\
|E(G)| &= \frac{1}{|V(G)| - 2} \sum_{v \in V(G)} |E(G - v)|.
\end{align*}
Reconstruction Example

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Reconstruction Example

\[ |V(G)| = 5. \]

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Reconstruction Example

$|V(G)| = 5.$

$|E(G)| = \frac{1}{3}(5 + 2 + 4 + 4 + 3).$
Reconstruction Example

\[ |V(G)| = 5. \]

\[ |E(G)| = \frac{1}{3} (5 + 2 + 4 + 4 + 3) = 6. \]
Reconstruction Example

\[ |V(G)| = 5. \]

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Reconstruction Example

\[ V(G) = 5. \]

\[ E(G) = 6. \]

Thus the degrees of the missing vertices in order are 1, 4, 2, 2, 3.
Reconstruction Example

Note the degree of the vertex deleted in the second graph is 4.
Reconstruction Example

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Thus it must have been connected to every other vertex:
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Determined Uniquely!
Recognizable Properties

Let $G$ have some property. This property is called **recognizable** if whenever we have a graph $H$ with the same deck as $G$, $H$ must also have the same property.
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- Since we can determine the degree sequence uniquely given a deck, regular graphs are recognizable.

A graph is **regular** if all its vertices have the same degree.
Class of reconstructible graphs

**Theorem** (Kelly, 1957) Regular graphs are reconstructible.

*Proof.*
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**Proof.**

Let $G$ be regular, and construct $D(G)$. 
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*Proof.*

- Let \( G \) be regular, and construct \( \mathcal{D}(G) \).
- By previous slide, the regularity of \( G \) is recognizable.
Class of reconstructible graphs

**Theorem** (Kelly, 1957) Regular graphs are reconstructible.

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- Let $G$ be regular, and construct $\mathcal{D}(G)$.
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- Add a vertex to any card.
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- By previous slide, the regularity of $G$ is recognizable.
- Let $d$ be the degree of each vertex.
- Add a vertex to any card.
- Connect new vertex to each of the $d$ vertices in the card with degree $d - 1$, and $G$ is reconstructed uniquely.
Kelly’s Lemma

\[ s(F, G) = \# \text{ of subgraphs of } G \text{ isomorphic to } F. \]
Kelly’s Lemma

- \( s(F, G) = \# \) of subgraphs of \( G \) isomorphic to \( F \).

- **Kelly’s Theorem:** For all \( F \) with less vertices than \( G \),

\[
s(F, G) = \frac{1}{(|V(G)| - |V(F)|)} \sum_{v \in V(G)} s(F, G - v).
\]
**Proof of Kelly’s Theorem:**

Double count ordered pairs of the form

\[(H, G - v)\]

where \(H \cong F\), and \(H \subset G - v\).
A Nice Consequence of Kelly’s Lemma

Let $F = K_2$ (edges). Then $s(K_2, G) = |E(G)|$, and...
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Thus, we have that Kelly’s Lemma implies that the number of edges is determined by the deck.
Even more

However, Kelly’s Lemma gives us even more: \( s(F, G) \) for any \( F \).
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Even more

- However, Kelly’s Lemma gives us even more: $s(F, G)$ **for any** $F$.

- Thus if there are two graphs with the same deck, they must have the same:
  - number of vertices.
  - number of edges.
  - number of subgraphs of EVERY type (same $s(F, G)$ for all $F$ with $|V(F)| < |V(G)|$).
Thoughts on the Conjecture

So maybe one would think then that all graphs are reconstructible, and the conjecture is true?
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- The conjecture has been confirmed after all for all graphs with less than 11 vertices (12.3 million graphs).
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- The conjecture has been confirmed after all for all graphs with less than 11 vertices (12.3 million graphs).

- However, there are good reasons to believe it false...
False Reconstruction Conjectures

- Digraphs are not reconstructible in general.
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- There are arbitrarily large counterexamples. (Stockmeyer, 1977/1981)
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False Reconstruction Conjectures

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- Hypergraphs are not, in general, reconstructible.
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- Infinite graphs are not, in general, reconstructible.
Reconstruction Numbers

- Related Topic: Graph Reconstruction Numbers.
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The existential reconstruction number of $G$, $\exists rn(G)$, is the minimum number of cards in $\mathcal{D}(G)$ required to reconstruct $G$ uniquely.
Reconstruction Numbers

- Related Topic: **Graph Reconstruction Numbers.**

- The **existential reconstruction number** of $G$, $\exists rn(G)$, is the minimum number of cards in $\mathcal{D}(G)$ required to reconstruct $G$ uniquely.

- The **universal reconstruction number** of $G$, $\forall rn(G)$, is the minimum number $n$ such that all multisubsets of $\mathcal{D}(G)$ of size $n$ uniquely reconstruct $G$. 
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- The **universal reconstruction number** of $G$, $\forall \text{rn}(G)$, is the minimum number $n$ such that all multisubsets of $\mathcal{D}(G)$ of size $n$ uniquely reconstruct $G$.

- Almost all graphs have $\exists \text{rn}(G) = 3$. (Bollobas, 1990).
The last three cards in this deck are as well in the deck of
Example

- The last three cards in this deck are as well in the deck of

- However, the first three graphs DO determine the graph $G$ uniquely.
Example

The last three cards in this deck are as well in the deck of

However, the first three graphs DO determine the graph $G$ uniquely.

Thus, for this $G$, $\exists \text{rn}(G) = 3$, $\forall \text{rn}(G) = 4$. 
Reconstruction Numbers

Found for all graphs on at most 10 vertices:

<table>
<thead>
<tr>
<th>∃rn</th>
<th>n = 3</th>
<th>n = 4</th>
<th>n = 5</th>
<th>n = 6</th>
<th>n = 7</th>
<th>n = 8</th>
<th>n = 9</th>
<th>n = 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>8</td>
<td>34</td>
<td>150</td>
<td>1044</td>
<td>12,334</td>
<td>274,666</td>
<td>12,005,156</td>
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<tr>
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</table>

Very hard to get any further: there are 1,018,997,864 graphs on 11 vertices!
Other Related Problems

- **The Legitimate Deck Problem:** Given a multiset of graphs $\mathcal{G}$, does there exist a graph $H$ such that $\mathcal{D}(H) = \mathcal{G}$?
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- **The problem of k-reconstruction**: Instead of deleting 1 vertex to form each card, delete \( k \) vertices. Then determine \( f(k) \), if it exists, such that all graphs with greater than \( f(k) \) vertices are \( k \)-reconstructible.
Other Related Problems

- **The Legitimate Deck Problem:** Given a multiset of graphs $\mathcal{G}$, does there exist a graph $H$ such that $\mathcal{D}(H) = \mathcal{G}$?

- **The problem of $k$-reconstruction:** Instead of deleting 1 vertex to form each card, delete $k$ vertices. Then determine $f(k)$, if it exists, such that all graphs with greater than $f(k)$ vertices are $k$-reconstructible.

- The Graph Reconstruction Conjecture says $f(1) = 2$. 

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References

Bollobás, B., Almost every graph has reconstruction number three. *J. Graph Theory* **14** (1990), 1–4.

