An Introduction To Ramsey Theory for Graphs

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Abstract

A graph is a set of vertices with some pairs of vertices connected by edges. Graphs are used to model a number of phenomena, from biological and physical real life problems, to number theoretic and other mathematical problems. The study of partitioning substructures of graphs has been studied quite extensively. Ramsey theory is often described as the study of preservation of structure under partitioning. In this talk, I will survey some of the classic Ramsey theory results, and survey specifically some results from Ramsey theory on graphs.
Graph Theory

The study of Graphs ...
The study of Graphs ... so what's a graph?
Graph Theory

We consider only the simplest types of graphs
Ramsey Theory

- Study of preservation of structure under partition.
Ramsey Theory

- Study of preservation of structure under *partition*. 
Ramsey Theory

- Study of preservation of structure under colouring.
Ramsey Theory

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- A structure is partitioned (coloured) in two classes. What kind of structure can be guaranteed within one of the two partition (colour) classes?
Ramsey Theory

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- Now generalize to $r$ partitions (colours).
Ramsey Theory

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- Now generalize to $r$ partitions (colours).

Think of the pigeon hole principle - two pigeons in one hole is the "structure" we are guaranteed.
Ramsey Theory Notation, Example

- "Arrow" notation $F \rightarrow (G)_r^H$ means...
Ramsey Theory Notation, Example

- "Arrow" notation $F \rightarrow (G)^H_r$ means...

- For all partitions of the $H$-substructures of $F$ into $r$ classes, there exists a $G$-substructure of $F$, with all its $H$-substructures in the same class.
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(see overhead)
Ramsey Theory Notation, Example

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Examples:

- $\{1, \ldots, n\} \rightarrow (2 \text{ numbers})_{n-1}^{\text{numbers}}$
Ramsey Theory Notation, Example

- "Arrow" notation $F \rightarrow (G)^H_r$

Examples:

- $\{1, \ldots, n\} \rightarrow (2 \text{ numbers})^{\text{numbers}}_{n-1}$
- $\{1, \ldots, 5\} \rightarrow (x, y, z \text{ such that } x + y = z)^{\text{numbers}}_2$
Ramsey Theory Notation, Example

- "Arrow" notation $F \to (G)_r^H$

Examples:

- $\{1, \ldots, n\} \to (2 \text{ numbers})_{n-1}^{\text{numbers}}$
- $\{1, \ldots, 5\} \to (x, y, z \text{ such that } x + y = z)_2^{\text{numbers}}$
- $\{1, \ldots, 9\} \to (\text{AP}_3)_2^{\text{numbers}}$
Ramsey Theory on Graphs

For graphs $G$, $H$ and $r \in \mathbb{Z}^+$,

$$F \rightarrow (G)_r^H$$

means that for every colouring of the $H$-subgraphs of $F$ in $r$ colours, there exists a copy of $G$ in $F$ with all of its $H$-subgraphs the same colour.
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Example:
Ramsey Theory on Graphs

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Example:

\[
\begin{array}{c}
\text{K}_6 \rightarrow \left( \begin{array}{c}
\text{triangle} \\
\end{array} \right)_2
\end{array}
\]
Ramsey Theory on Graphs

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means that for every colouring of the $H$-subgraphs of $F$ in $r$ colours, there exists a copy of $G$ in $F$ with all of its $H$-subgraphs the same colour.

- Example:
Ramsey Theory on Graphs

Question: How small can a graph be, and stillarrow the given graph $G$?
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The Ramsey number $R(k)$ is the smallest $n$ such that

$$K_n \rightarrow (K_k)_2$$
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The Ramsey number $R(k)$ is the smallest $n$ such that

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Example (from previous page):

$$K_6 \to (K_3)_2$$
Ramsey Theory on Graphs

- Question: How small can a graph be, and still arrow the given graph $G$?
- The Ramsey number $R(k)$ is the smallest $n$ such that
  \[ K_n \rightarrow (K_k)_2 \]
- Example (from previous page):
  \[ K_6 \rightarrow (K_3)_2 \]
- This shows that $R(3) \leq 6$. (Easy exercise to show $R(3) = 6$)
Ramsey Numbers

Known values for the Ramsey numbers:
Ramsey Numbers

Known values for the Ramsey numbers:

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\[ 43 \leq R(5) \leq 49 \]
Ramsey Numbers

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\[ 43 \leq R(5) \leq 49 \]
\[ 102 \leq R(6) \leq 165 \]
Ramsey Numbers

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\[ 102 \leq R(6) \leq 165 \]

- Ramsey numbers are hard to find (even the small values!)
Sparse Ramsey Graphs: $K_k$-free

Is there a $K_6$-free graph $F$ such that $F \rightarrow (K_3)^2$?
Sparse Ramsey Graphs: $K_k$-free

Is there a $K_6$-free graph $F$ such that

$$F \rightarrow (K_3)^2$$

YES!
Sparse Ramsey Graphs: $K_k$-free

Is there a $K_6$-free graph $F$ such that

$$F \rightarrow (K_3)_2 ?$$

YES!

Found in 1968.
Is there a $K_5$-free graph $F$ such that

$$F \rightarrow (K_3)_2?$$
Sparse Ramsey Graphs: $K_k$-free

Is there a $K_5$-free graph $F$ such that

$$F \rightarrow (K_3)_2$$

YES! Found in 1999.
15 vertices suffices.
659 distinct graphs found.
This IS minimum.
Is there a $K_4$-free graph $F$ such that $F \rightarrow (K_3)_2$?
Sparse Ramsey Graphs: $K_k$-free

Is there a $K_4$-free graph $F$ such that

$$F \rightarrow (K_3)_2?$$

3,000,000,000 vertices!
Probabilistic
(existence theorem only).
Nothing better known!
Sparse Ramsey Graphs: $K_k$-free

- Is there a $K_4$-free graph $F$ such that
  \[ F \rightarrow (K_3)_2? \]

- So, the graphs don’t even have to be very dense with edges at all!
Sparse Ramsey Graphs: $K_k$-free

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- Other results have shown that we can find a graph that works with pretty much any property we want.
Conclusion
Thanks for coming!