# Extremal Graph Theory

Can be viewed as the study of how graph constants ensure certain properties

| Graph Constant $\implies$ | Property |
|:---:|:---:|
| order | contains $K_3$ |
| size | contains $K_m$ |
| connectivity | contains $G$ |
| min degree | contains a cycle |
| max degree | is $r$-colourable |
| $\chi(G)$ | is $k$-partite |
| diameter | |

For a first example,

**Theorem (Mantel's Theorem (1905)).**
For $n \geq 3$, if a graph $G$ with $n$ vertices has more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, then $G$ contains a triangle (a subgraph isomorphic to $K_3$).

Note this is best possible, since the graph

$$K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$$

has $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges, and does not contain a triangle.

In general, for any $n \in \mathbb{Z}^+$, and any graph $G$, let

$$ex(n; G)$$

denote the largest integer $e$ (if any exists) such that there exists a graph on $n$ vertices with $e$ edges not containing a subgraph isomorphic to $G$.

The number $ex(n; G)$ is called an **extremal number** (or sometimes a **Turan number**, for reasons we will see shortly).

A graph $F$ on $n$ vertices is called an **extremal graph for** $G$ if $F$ contains no copy of $G$, and $|E(F)| = ex(n; G)$.

Fix a graph $G$, and $n, e \in \mathbb{Z}^+$.

To show that $ex(n; G) \geq e$, it is necessary and sufficient to show there exists a graph $F$ on $n$ vertices with $e$ edges such that $F$ does not contain a copy of $G$.

To show that $ex(n; G) < e$, it is necessary and sufficient to show that EVERY graph with $n$ vertices and $e$ edges contains a copy of $G$ (or equivalently that if a graph $F$ on $n$ vertices does not contain a copy of $G$, then $|E(F)| < e$).

For any $n, k \in \mathbb{Z}^+$, let $T(n, k)$ denote the $k$-partite graph with $n$ vertices where the partite sets are as equal in size as possible.

For example, $T(12, 5) = \ldots$

Note that if $n < k$, then $T(n, k) = K_n$.

Let $t(n, k) = |E(T(n, k))|$.

**Proposition.**

$$t(n, k) = \binom{k}{2} + (n - k)(k - 1) + t(n - k, k).$$

**Theorem (Turán's Theorem (1941/1955)).**
Let $n, k \in \mathbb{Z}^+$. Then $ex(n; K_{k+1}) = t(n, k)$ and $T(n, k)$ is the unique extremal $K_{k+1}$-free graph.

**Proof...**

The extremal graph for $K_3$ is bipartite. If we look at non-bipartite graphs, we can shave off many edges.

**Theorem (Erdős-Gallai (1962)).** For any non-bipartite graph $G$ with $n$ vertices, if

$$E(G) \geq ex(n - 1; K_3) + 2 = \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + 2,$$

then $G$ contains a triangle.

**Theorem (not sure who did this).** Let $V$ be a set with $|V| = 2n$, and let $G$ be a graph with vertex set $V$. Then $G$ contains a bipartite subgraph $H$ in which each partite set has size $n$, and

$$|E(H)| \geq \frac{1}{4}|E(G)|.$$

Generalizing to hypergraphs...

**Theorem.** Let $V$ be a set with $|V| = rn$, and let $G$ be an $r$-uniform hypergraph with vertex set $V$. Then $G$ contains a $r$-partite $r$-uniform subhypergraph $H$ in which each partite set has size $n$, and

$$|E(H)| \geq \frac{r!}{r^r}|E(G)|.$$

# Ramsey Theory

Can be viewed as the study of "structure" under partition (or colouring).



"Complete disorder is impossible" — T. S. Motzkin

"Of three ordinary people, two must have the same sex."
—D. J. Kleitman.

The concept of a partition, or a colouring, is crucial to Ramsey theory.

If $S$ and $C$ are sets with $|C| = r \geq 2$, any function $f : S \to C$ is an *r-colouring* of $S$, and the elements of $C$ are called *colours*. For each $i \in C$, $f^{-1}(i) \subseteq S$ is called a *colour class*, and any subset of a colour class is said to be *monochromatic*. For example, if $S = \{a, b, c\}$, $C = \{\text{red}, \text{blue}\}$, and $f(a) = f(c) = \text{red}$, and $f(b) = \text{blue}$, then the colour classes are $\{a, c\}$ and $\{b\}$.

The pigeonhole principle is a basic tool in Ramsey theory, and is itself now considered the simplest "Ramsey-type" theorem.

**Theorem (Pigeonhole principle).**
If at least $mr + 1$ objects are partitioned into $r$ (possibly empty) subsets, at least one subset contains $m + 1$ elements.

The pigeonhole principle can be restated in various ways, *e.g.*:

(a) If $S$ is a set with $|S| \geq mr + 1$, and $S = S_1 \cup \cdots \cup S_r$ is a partition of $S$, then $\exists\, i \in [1, r]$ such that $|S_i| \geq m + 1$.

(b) If $f : [1, mr + 1] \rightarrow [1, r]$, then $\exists\, i \in [1, r]$ such that $|f^{-1}(i)| \geq m + 1$.

**Theorem (Infinite pigeonhole principle).**
$\forall$ finite colouring of an infinite set, there exists a colour class which is infinite.

Two theorems due to Ramsey generalize the finite and infinite versions of the pigeonhole principle to the colouring of $k$-sets, rather than just singletons.

**Theorem (Ramsey's theorem for finite sets, 1930).**
$\forall\ m, k, r \in \mathbb{Z}^+$, $\exists$ a least $n = R_k(m; r) \in \mathbb{Z}^+$ such that $\forall\ n$-set $N$, and $\forall\ r$-colouring $f : [N]^k \to [1, r]$ of the $k$-subsets of $N$, $\exists$ $M \in [N]^m$ such that $[M]^k$ is monochromatic.

**Theorem (Ramsey's theorem for infinite sets, 1930).**
$\forall\ k, r \in \mathbb{Z}^+$, every infinite set $X$, and every $f : [X]^k \to [1, r]$, $\exists$ an infinite set $Y \subseteq X$ such that $[Y]^k$ is monochromatic.

When $k = 1$, the two versions of Ramsey's theorem are exactly the two versions of the pigeonhole principle.

To show that $R_k(m; r) > x$, it suffices to exhibit an $r$-colouring of the $k$-tuples of an $x$-set containing no $m$-set $M$ such that $[M]^k$ is monochromatic.

To show that $R_k(m; r) \leq x$, it suffices to prove that for every $r$-colouring of the $k$-tuples of an $x$-set, there is an $m$-set $M$ such that $[M]^k$ is monochromatic.

It is a fairly quick proof (a good exercise) to see that if Ramsey's theorem is true for $r = 2$, then it is true for all $r$. So we will prove it in the case $r = 2$.

In order to prove Ramsey's theorem, we will introduce an off-diagonal version. Let $R_k(a, b)$ be the least integer $n$ such that $\forall$ $n$-set $N$, and $\forall$ $r$-colouring $f : [N]^k \to \{\text{red}, \text{blue}\}$ of the $k$-subsets of $N$, $\exists$ either $M_1 \in [N]^a$ such that $[M_1]^k$ is monochromatic red, or a $M_2 \in [N]^b$ such that $[M_2]^k$ is monochromatic blue.

Note that:
$$R_k(a, b) \leq R_k(\max\{a, b\}; 2),$$
$$R_k(m; 2) = R_k(m, m).$$

An "arrow notation" (known as a "Ramsey arrow") is used to simplify statements that are similar to Ramsey's theorem. (introduced by Erdős and Rado in 1953).

For positive integers $n, m, k$ and $r$, write

$$n \longrightarrow (m)^k_r$$

if $\forall$ $n$-set $N$, and $\forall$ $f : [N]^k \rightarrow [1, r]$, $\exists$ $M \in [N]^m$ such that $[M]^k$ is monochromatic.

Ramsey-type theorems can be hard to read and understand at first due to the number of alternating quantifiers. For example, there are a total of four quantification switches in Ramsey's theorem (stated below for colouring the integers):

$$\forall m, k, r \in \mathbb{Z}^+, \exists n \in \mathbb{Z}^+ \text{ s.t. } \forall f : [1, n]^k \rightarrow [1, r],$$
$$\exists i \in [1, r] \text{ and } S \in [1, n]^m \text{ s.t. } \forall S' \in [S]^k, f(S') = i.$$

Using the arrow notation, Ramsey's theorem can be restated briefly:

**Theorem (Ramsey's theorem restated).**
$\forall\, m, k, r \in \mathbb{Z}^+$, $\exists$ a least integer $n = R_k(m; r)$ such that

$$n \longrightarrow (m)_r^k.$$

GRAPH RAMSEY THEORY

The arrow notation has been generalized to graphs as follows: let $F$ be a "large" graph, $G$ be a "medium sized" graph, and $H$ a "small" graph.

A "copy" of $H$ in $G$ means a subgraph (either weak or induced depending on context) of $G$ isomorphic to $H$. If $\forall$ $r$-colouring of the copies of $H$ in $F$, there is a copy $G'$ of $G$ in $F$ such that every copy of $H$ in $G'$ is the same colour, then write

$$F \longrightarrow (G)_r^H.$$

The field of "Graph Ramsey theory" is essentially the study of aspects of various graph Ramsey arrows. Since the copies of $K_k$ inside $K_n$ are in one-to-one correspondence with the $k$-subsets of an $n$-set, Ramsey's theorem can be restated in terms of graphs.

**Theorem (Ramsey's theorem—graph theoretic).**
$\forall\, m, k, r \in \mathbb{Z}^+$, $\exists$ a least integer $n = R_k(m; r)$ such that

$$K_n \longrightarrow (K_m)_r^{K_k}.$$

Hilbert's Cube lemma is one of the earliest examples of a Ramsey-type result, predating Ramsey himself, dating from 1892.

Fix $x_0 \in \mathbb{Z}^+ \cup \{0\}$, $d \in \mathbb{Z}^+$, and let $x_1, \ldots, x_d \in \mathbb{Z}^+$. The family of integers

$$H(x_0, x_1, \ldots, x_d) = \left\{ x_0 + \sum_{i \in I} x_i : I \subseteq [1, d] \right\}$$

is called an affine cube of dimension $d$ (or an affine $d$-cube). For example,

$$H(0, 1, 1, 1) = \{0, 1, 2, 3\} \quad \text{and} \quad H(5, 2, 3) = \{5, 7, 8, 10\}.$$

**Theorem (Hilbert, 1892).**
$\forall \ r, d \in \mathbb{Z}^+$, $\exists \ n = h(d; r)$ such that $\forall \ r$-colouring $f : [1, n] \to [1, r]$, $\exists$ a monochromatic affine $d$-cube.

An **arithmetic progression of length $k$ with difference** $b$, written $AP_k$, is a sequence of numbers of the form $\{a, a+b, a+ 2b, \cdots, a+(k-1)b\}$. For example,

$$3, 8, 13, 18$$

is an $AP_4$ with difference 5.

**Theorem (van der Waerden, 1927).**
Fix $k, r \in \mathbb{Z}^+$. Then $\exists\, n = W(k, r)$ such that for every $r$-colouring $f : [1, n] \to [1, r]$, $\exists$ a monochromatic $AP_k$ (arithmetic progression of length $k$). *i.e.*, $\exists\, a, d \in \mathbb{Z}^+$, $c \in [1, r]$ such that $\{a + id : i \in [1, k]\} \subseteq [1, n]$, and for all $i \in [1, k]$, $f(a + id) = c$.

The bounds on $W(k, r)$ produced by the original proof to van der Waerden's theorem are very poor, and still today little is known.

Related to van der Waerden's theorem...

**Theorem (Schur's Lemma, 1916).**
Fix $r \in \mathbb{Z}^+$. Then $\exists$ a least integer $n = S(r)$ such that for every $r$-colouring $f : [1, n] \to \{1, 2, \ldots, r\}$, there exist $x, y \in [1, n]$ such that $f(x) = f(y) = f(x + y)$.

**Proof...**

Fix $r \in \mathbb{Z}^+$, and let $n = R_2(3; r) - 1$. Let $f : [1, n] \to [1, r]$ be an $r$-colouring of $[1, n]$. Define an $r$-colouring $f'$,

$$f' : [1, n+1]^2 \to [1, r],$$

where for $a, b \in [1, n+1]$, $a < b$, $f'(\{a, b\}) = f(b - a)$. Then by the choice of $n$, $\exists$ a triangle, monochromatic under $f'$. i.e., $\exists$ $u, v, w \in [1, n+1]$, $u < v < w$ such that

$$f'(\{u, v\}) = f'(\{u, w\}) = f'(\{v, w\}).$$

By the definition of $f'$, it then follows that

$$f(v - u) = f(w - u) = f(w - v).$$

Let $x = v - u$, $y = w - v$. Then $f(x) = f(y) = f(x + y)$. $\qquad \square$

**Corollary.**
For all $r \in \mathbb{Z}^+$, $S(r) \le R_2(3; r) - 1$.