Undergraduate Research in Orthogonal Matrices*

R. Craigen

Summer 2006

Abstract

Consider the following four (partitioned) 4×4 (± 1)-matrices:

$$\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}; \quad \left(\begin{array}{c|ccc} 1 & 1 & 1 & 1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right); \quad \left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ \hline 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{array}\right); \quad \left(\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array}\right)$$

They share some properties: in each one, any two rows match in the exactly half the positions. Square (± 1)-matrices with this property are called *Hadamard matrices*—the most celebrated class of *orthogonal matrices* (the more general objects are characterized by various other restrictions on entries and variations on the equation $XX^T = \lambda I$).

In other ways they differ: Look for some interesting structure within and between the blocks formed by the partition lines in our examples. Try building 8×8 Hadamard matrices that exhibit similar structures. Can you see how the last one is self-similar in a certain way? This leads to an important infinite class of Hadamard matrices having many applications.

Hadamard's 1893 conjecture that the matrices named in his honour exist in every order of the form $n = 4k$ remains unresolved today, and is regarded as one of the two most important problems in the field of combinatorics (the other deals with projective planes). Exactly 100 years later I proved an asymptotic existence result that, to this day, is the closest, in one sense, that we have come to settling this question; this summer we will chip away at it a bit more.

We will examine the fascinating inner structure of Hadamard (and other orthogonal) matrices, some of which is hinted at in our examples above. It has long been my conviction that understanding this structure will be the key to solving this and many related open questions. Accordingly I have many small projects in which a microscope is turned on various aspects of structure, including some dandy puzzles for aspiring researchers.

We will also look at some questions regarding the classification of these objects and cook up some great recipes for building them, subject to additional specifications (as demanded by various applications or theoretical considerations).

The matrices we study are used in "error-correction codes" (used for CD's and DVD's, wi-fi, etc.), range-finding devices (like GPS), optical filtering, interferometry, discrete Fourier analysis and many other areas. Recently some important questions relating to cutting-edge technological development in quantum computing, quantum information and quantum learning have been reduced to some the very "purely theoretical" questions about orthgonal matrices that we will be addressing.

*Warning: research in this field may be addictive! It involves exposure to dangerous levels of beautiful objects and tantalizing questions.