§21.3 Nonhomogeneities

In Section 21.2 we stressed the fact that separation of variables is carried out on **linear**, **homogeneous** PDEs and **linear**, **homogeneous** boundary and/or initial conditions. Separated functions are superposed in order to satisfy nonhomogeneous initial conditions. When nonhomogeneities are present in the PDE, or in the boundary conditions of time-dependent problems, separation by itself fails. To illustrate, reconsider vibration problem 21.9 for displacement of a taut string with fixed end points, but with gravity taken into account:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0, \quad (g = 9.81), \tag{21.33a}$$

$$y(0,t) = 0, \quad t > 0, \tag{21.33b}$$

$$y(L,t) = 0, \quad t > 0, \tag{21.33c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{21.33d}$$

$$y_t(x,0) = 0, \quad 0 < x < L. \tag{21.33e}$$

Only the partial differential equation is affected; it is nonhomogeneous. The boundary conditions remain homogeneous. Substitution of a separated function $y(x,t) = X(x)T(t)$ into PDE 21.33a gives

$$XT'' = c^2 X''T - g.$$

Our usual procedure of dividing by $X(x)T(t)$ would not lead to a separated equation; in fact, this equation cannot be separated. Likewise, were boundary condition 21.33b not homogeneous, say $y(0,t) = f(t)$, in which case the left end of the string would be forced to undergo specific motion, substitution of $y(x,t) = X(x)T(t)$ would not lead to information about $X(x)$ and $T(t)$ separately.

In this section we illustrate one of two methods for handling nonhomogeneities. The method uses steady-state solutions for heat conduction problems and static deflections for vibration problems. It applies, however, only to time-independent nonhomogeneities. The other method handles time-dependent nonhomogeneities, but it is a much more involved procedure.

Time-Independent Nonhomogeneities

Partial differential equation 21.33a has a time-independent nonhomogeneity (it is also independent of x, but that is immaterial). To solve this problem, we define a new dependent variable $z(x,t)$ according to

$$y(x,t) = z(x,t) + \psi(x), \tag{21.34}$$

where $\psi(x)$ is the solution of the corresponding **static-deflection problem**

$$0 = c^2 \frac{d^2\psi}{dx^2} - g, \quad 0 < x < L, \tag{21.35a}$$

$$\psi(0) = 0, \quad \psi(L) = 0. \tag{21.35b}$$

It is obtained from problem 21.33 by removing all time dependence and calling the solution $\psi(x)$ rather than $y(x)$. Its solution, called the **static deflection solution**, is the shape of the string were it to lie motionless under the influence of its internal tension and gravity. Differential equation 21.35a implies that

$$\psi(x) = \frac{gx^2}{2c^2} + Ax + B,$$

and boundary conditions 21.35b require

$$0 = B, \qquad 0 = \frac{gL^2}{2c^2} + AL + B.$$

From these we obtain the position of the string were it to hang motionless under gravity:

$$\psi(x) = \frac{gx}{2c^2}(x - L). \tag{21.36}$$

We expect that the string will vibrate about this position and that $z(x,t)$ represents displacements from this position. A PDE satisfied by $z(x,t)$ can be found by substituting representation 21.34 into PDE 21.33a:

$$\frac{\partial^2}{\partial t^2}[z(x,t) + \psi(x)] = c^2 \frac{\partial^2}{\partial x^2}[z(x,t) + \psi(x)] - g.$$

This equation simplifies to the following homogeneous PDE when we note that $\psi(x)$ is only a function of x that satisfies ODE 21.35a:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \tag{21.37a}$$

Boundary conditions for $z(x,t)$ are obtained by setting $x = 0$ and $x = L$ in representation 21.34 and using boundary conditions 21.33b,c for $y(x,t)$:

$$z(0,t) = y(0,t) - \psi(0) = 0, \quad t > 0, \tag{21.37b}$$

$$z(L,t) = y(L,t) - \psi(L) = 0, \quad t > 0. \tag{21.37c}$$

Finally, by setting $t = 0$ in 21.34 and its partial derivative with respect to t, and using initial conditions 21.33d,e for $y(x,t)$, we obtain initial conditions for $z(x,t)$:

$$z(x,0) = y(x,0) - \psi(x) = f(x) + \frac{gx}{2c^2}(L - x), \quad 0 < x < L, \tag{21.37d}$$

$$z_t(x,0) = y_t(x,0) = 0, \quad 0 < x < L. \tag{21.37e}$$

We have therefore replaced problem 21.33, which has a nonhomogeneous PDE, with problem 21.37, which has a homogeneous PDE. We have complicated one of the initial conditions, but this is a small price to pay. If a function $z(x,t) = X(x)T(t)$ with variables separated is to satisfy PDE 21.37a, boundary conditions 21.37b,c, and initial condition 21.37e, then $X(x)$ and $T(t)$ separately satisfy

$$X'' + \lambda X = 0, \quad 0 < x < L, \qquad\qquad T'' + \lambda c^2 T = 0, \quad t > 0,$$
$$X(0) = 0 = X(L); \qquad\qquad T'(0) = 0.$$

The Sturm-Liouville system was discussed in Section 19.2. According to line 1 of Table 19.1, eigenvalues are $\lambda_n = n^2\pi^2/L^2$ and corresponding eigenfunctions are $X_n(x) = \sin(n\pi x/L)$. Since the auxiliary equation for the differential equation in $T(t)$ is $m^2 + c^2\lambda_n = 0$, with solution $m = \pm c\sqrt{\lambda_n}i = \pm n\pi ci/L$, a general solution of the differential equation is $T(t) = A\cos\frac{n\pi ct}{L} + B\sin\frac{n\pi ct}{L}$. The condition $T'(0) = 0$ requires $B = 0$. Separated functions are therefore $b\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$. Because PDE 21.37a and conditions 21.37b,c,e are linear and homogeneous, we superpose these functions and take

$$z(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}. \tag{21.38}$$

Initial condition 21.37d requires the constants b_n to satisfy

$$f(x) + \frac{gx}{2c^2}(L - x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

Consequently, the b_n are coefficients in the Fourier sine series of the odd extension of $f(x) + gx(L-x)/(2c^2)$ to a function of period $2L$; that is,

$$b_n = \frac{2}{L}\int_0^L \left[f(x) + \frac{gx}{2c^2}(L-x)\right]\sin\frac{n\pi x}{L}dx. \tag{21.39}$$

The formal solution of vibration problem 21.33 is therefore

$$y(x,t) = \frac{gx}{2c^2}(x-L) + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \tag{21.40}$$

where the b_n are given by the integral in equation 21.39.

This technique of spliting off static deflections can be applied to any nonhomogeneity that is only a function of position, be it in the PDE or in a boundary condition. We illustrate nonhomogeneities in boundary conditions in the next example.

Example 21.4 Solve the initial boundary value problem for temperature in a homogeneous, isotropic rod with insulated sides when the ends of the rod are held at constant temperatures,

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{21.41a}$$

$$U(0,t) = U_0, \quad t > 0, \tag{21.41b}$$

$$U(L,t) = U_L, \quad t > 0, \tag{21.41c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{21.41d}$$

Solution We define a new dependent variable $V(x,t)$ by

$$U(x,t) = V(x,t) + \psi(x), \tag{21.42}$$

where $\psi(x)$ is the solution of the associated **steady-state problem**

$$0 = k\frac{d^2\psi}{dx^2}, \quad 0 < x < L, \tag{21.43a}$$

$$\psi(0) = U_0, \tag{21.43b}$$

$$\psi(L) = U_L. \tag{21.43c}$$

It is obtained from problem 21.41 by removing all time dependence and calling the solution $\psi(x)$ rather than $U(x)$. Its solution, called the **steady-state solution**, is anticipated to be the temperature in the rod after a very long time. Differential equation 21.43a implies that $\psi(x) = Ax + B$, and boundary conditions 21.43b,c require

$$U_0 = B, \qquad U_L = AL + B.$$

From these, we obtain the steady-state solution

$$\psi(x) = U_0 + \frac{(U_L - U_0)x}{L}. \tag{21.44}$$

With this choice for $\psi(x)$, the PDE for $V(x,t)$ can be found by substituting representation 21.42 into 21.41a:

$$\frac{\partial}{\partial t}[V(x,t) + \psi(x)] = k\frac{\partial^2}{\partial x^2}[V(x,t) + \psi(x)].$$

Because $\psi(x)$ is only a function of x that has a vanishing second derivative, this equation simplifies to

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0. \tag{21.45a}$$

Boundary conditions for $V(x,t)$ are obtained from representation 21.42 and boundary conditions 21.41b,c for $U(x,t)$:

$$V(0,t) = U(0,t) - \psi(0) = U_0 - U_0 = 0, \quad t > 0, \tag{21.45b}$$
$$V(L,t) = U(L,t) - \psi(L) = U_L - U_L = 0, \quad t > 0. \tag{21.45c}$$

Finally, $V(x,t)$ must satisfy the initial condition

$$V(x,0) = U(x,0) - \psi(x) = f(x) - U_0 - \frac{(U_L - U_0)x}{L}, \quad 0 < x < L. \tag{21.45d}$$

Separation of variables $V(x,t) = X(x)T(t)$ on 21.45a,b,c leads to the ordinary differential equations

$$X'' + \lambda X = 0, \quad 0 < x < L, \quad (21.46a) \qquad T' + k\lambda T = 0, \quad t > 0. \quad (21.47)$$
$$X(0) = X(L) = 0; \quad (21.46b)$$

The Sturm-Liouville system was discussed in Section 19.2. According to line 1 of Table 19.1, eigenvalues are $\lambda_n = n^2\pi^2/L^2$ and corresponding eigenfunctions are $X_n(x) = \sin(n\pi x/L)$. Since the auxiliary equation for the differential equation in $T(t)$ is $m + k\lambda_n = 0$, with solution $m = -k\lambda_n$, a general solution of the differential equation is $T(t) = be^{-k\lambda_n t} = be^{-n^2\pi^2 kt/L^2}$. Separated functions are $be^{-n^2\pi^2 kt/L^2} \sin\dfrac{n\pi x}{L}$. Because the PDE and boundary conditions are linear and homogeneous, we superpose separated functions in the form

$$V(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}. \tag{21.48}$$

Initial condition 21.45d requires the constants b_n to satisfy

$$f(x) - U_0 - \frac{(U_L - U_0)x}{L} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad 0 < x < L.$$

Consequently, the b_n are the coefficients in the Fourier sine series of the odd extension of $f(x) - U_0 - (U_L - U_0)x/L$ to a function of period $2L$:

$$b_n = \frac{2}{L}\int_0^L \left[f(x) - U_0 - \frac{(U_L - U_0)x}{L}\right] \sin\frac{n\pi x}{L} dx. \tag{21.49}$$

The formal solution of problem 21.41 is therefore

$$U(x,t) = V(x,t) + U_0 + \frac{(U_L - U_0)x}{L}, \tag{21.50}$$

where $V(x,t)$ is given by the series in 21.48 and b_n by the integral in 21.49. Function $V(x,t)$ represents the transient part of the temperature function, which, because of the exponential factor $e^{-n^2\pi^2 kt/L^2}$, decreases with time. Temperature approaches the steady-state solution.●

It is interesting and informative to analyze solution 21.50 further for two specific initial temperature distributions $f(x)$. First, suppose that the initial temperature of the rod is 0°C throughout; that is, $f(x) \equiv 0$. In this case, equations 21.48–21.50 yield, for the temperature in the rod,

$$U(x,t) = U_0 + \frac{(U_L - U_0)x}{L} + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L}\int_0^L \left[-U_0 - (U_L - U_0)\frac{x}{L}\right] \sin\frac{n\pi x}{L} dx = \frac{-2}{n\pi}[U_0 + (-1)^{n+1}U_L].$$

This function is plotted for various fixed values of t in Figure 21.12 (using a diffusivity of $k = 12.4 \times 10^{-6}$ m^2/s). What is important to notice is the smooth transition from initial temperature 0°C to final steady-state temperature at every point in the rod except for its ends $x = 0$ and $x = L$. Here the transition is instantaneous, as is dictated by problem 21.41 when $f(x)$ is chosen to vanish identically. Physically, this is an impossibility, but the mathematics required to describe a very quick but smooth change in temperature from 0°C at $x = 0$ and $x = L$ to U_0 and U_L would complicate the problem enormously. In practice, then, we are willing to live with the anomaly of the solution at time $t = 0$ for $x = 0$ and $x = L$ in order to avoid these added complications. This anomaly is manifested in the heat transfer across the ends of the rod at time $t = 0$. According to equation 20.1 in Section 20.1, the amount of heat flowing left to right through any cross section of the rod is

$$q(x,t) = -\kappa\frac{\partial U}{\partial x} = -\kappa\left(\frac{U_L - U_0}{L} + \frac{\pi}{L}\sum_{n=1}^{\infty} nb_n e^{-n^2\pi^2 kt/L^2}\cos\frac{n\pi x}{L}\right)$$
$$= \frac{\kappa}{L}\left\{U_0 - U_L + 2\sum_{n=1}^{\infty}[U_0 + (-1)^{n+1}U_L]e^{-n^2\pi^2 kt/L^2}\cos\frac{n\pi x}{L}\right\}.$$

The series in this expression diverges (to infinity) when $x = 0$ at $t = 0$. In other words, the instantaneous temperature change at time $t = 0$ from 0°C to U_0°C is predicated on an infinite heat flux at that time. A similar situation occurs at the end $x = L$.

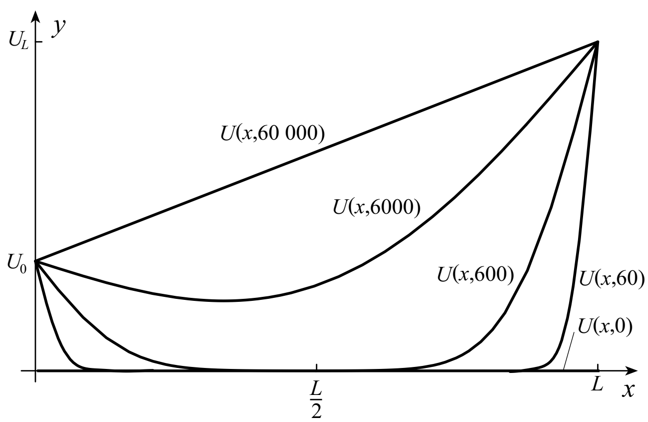

Figure 21.12

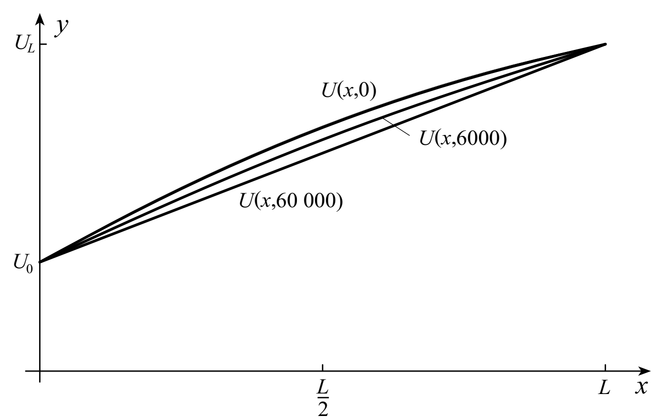

Figure 21.13

The second initial temperature function we consider is $f(x) = U_0(1 - x^2/L^2) + U_L x/L$, a distribution that does not give rise to abrupt temperature changes at time $t = 0$ since $f(0) = U_0$ and $f(L) = U_L$. In this case, coefficients b_n in 21.49 are $b_n = 4U_0[1 + (-1)^{n+1}]/(n^3\pi^3)$, and

$$U(x,t) = U_0 + \frac{(U_L - U_0)x}{L} + \frac{8U_0}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)^2\pi^2 kt/L^2}\sin\frac{(2n-1)\pi x}{L}.$$

As shown in Figure 21.13, the transition from initial to steady-state temperature is smooth for all $0 \le x \le L$. Supporting this is the heat flux vector

$$q(x,t) = \frac{\kappa}{L}\left[U_0 - U_L - \frac{8U_0}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2 kt/L^2}\cos\frac{(2n-1)\pi x}{L}\right].$$

The series herein converges for $0 \le x \le L$ and $t \ge 0$. If we take limits as $x \to 0^+$ and $t \to 0^+$, we find the initial heat flux across the end $x = 0$,

$$q(0+,0+) = \frac{\kappa}{L}\left[U_0 - U_L - \frac{8U_0}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\right] = \frac{\kappa}{L}\left[U_0 - U_L - \frac{8U_0}{\pi^2}\left(\frac{\pi^2}{8}\right)\right] = -\frac{\kappa U_L}{L}$$

(since $\sum_{n=1}^{\infty} 1/(2n-1)^2 = \pi^2/8$). Perhaps unexpectedly, we find that the direction of heat flow across $x = 0$ at time $t = 0$ is completely dictated by the sign of U_L. When $U_L < 0$, heat flows into the rod, and when $U_L > 0$, heat flows out. This is most easily seen by calculating the derivative of the initial temperature distribution in the rod at $x = 0$, $f'(0) = U_L/L$. If $U_L < 0$, points in the rod near $x = 0$ have temperature less than those in the end $x = 0$, and heat flows into the rod; if $U_L > 0$, points near $x = 0$ are at a higher temperature than those at $x = 0$, and heat flows out of the rod.

We have considered time-independent nonhomogeneities for initial boundary values problems associated with the one-dimensional heat and vibration equations, nonhomogeneites that can occur in the PDE and/or the boundary conditions. Boundary value problems associated with Laplace's equation in two dimensions

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad (21.51)$$

are somewhat different. When a nonhomogeneity $F(x, y)$ is introduced into this equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x, y), \qquad (21.52)$$

the equation is known as **Poisson's equation**. We shall not consider it here. What about nonhomogeneous boundary conditions? For instance, suppose we are to solve Laplace's equation in the rectangle shown to the right with nonzero boundary condition on all four sides. Unlike the heat and wave equations, we can break this problem into two homogeneous problems whose sum is the required function. The boundary value problem for $V(x, y)$ is

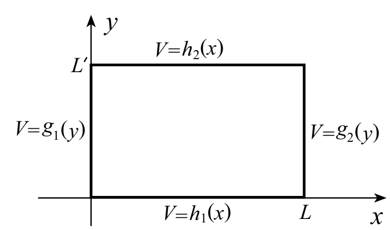

Figure 21.14

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L', \qquad (21.53a)$$

$$V(0, y) = g_1(y), \quad 0 < y < L', \qquad (21.53b)$$

$$V(L, y) = g_2(y), \quad 0 < y < L', \qquad (21.53c)$$

$$V(x, 0) = h_1(x), \quad 0 < x < L, \qquad (21.53d)$$

$$V(x, L') = h_2(x), \quad 0 < x < L. \qquad (21.53e)$$

Suppose that $V_1(x, y)$ and $V_2(x, y)$ are solutions of Laplace's equation satisfying the boundary conditions in Figure 21.15a,b. It is straightforward to show that the sum of these functions $V(x, y) = V_1(x, y) + V_2(x, y)$ satisfies problem 21.53. In addition, each of these problems can be solved by separation of variables. Here is an example.

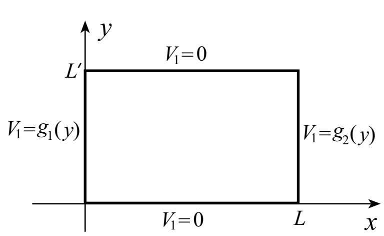

Figure 21.15a

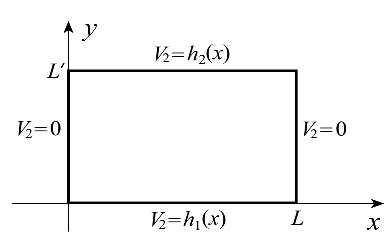

Figure 21.15b

Example 21.5 Find the solution of boundary value problem 21.53 when $g_1(y) = -1$, $g_2(y) = 0$, $h_1(x) = 1$, and $h_2(x) = 0$.

Solution The solution is the sum of the functions $V_1(x, y)$ and $V_2(x, y)$ that satisfy Laplace's equation inside the rectangles and the boundary conditions on the edges of the rectangles in Figures 21.16a,b. We solved the problem for $V_2(x, y)$ in Example 21.2. The solution is

$$V_2(x, y) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \left[\frac{e^{(2n-1)\pi(L'-y)/L} - e^{-(2n-1)\pi(L'-y)/L}}{e^{(2n-1)\pi L'/L} - e^{-(2n-1)\pi L'/L}} \right] \sin \frac{(2n-1)\pi x}{L}.$$

Solution $V_1(x, y)$ can be obtained by interchanging variables in $V_2(x, y)$ and reversing the sign,

$$V_1(x, y) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \left[\frac{e^{(2n-1)\pi(L-x)/L'} - e^{-(2n-1)\pi(L-x)/L'}}{e^{(2n-1)\pi L/L'} - e^{-(2n-1)\pi L/L'}} \right] \sin \frac{(2n-1)\pi y}{L'}.$$

The solution of the required problem is then $V_1(x, y) + V_2(x, y)$.●

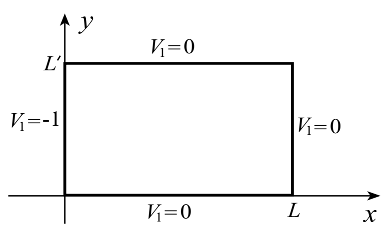

Figure 21.16a

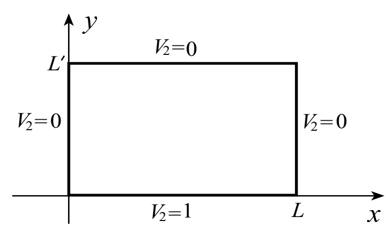

Figure 21.16b

EXERCISES 21.3

Part A Heat Conduction Problems

1. A homogeneous, isotropic rod with insulated sides has initial temperature distribution $U_L x/L$, $0 \le x \le L$ (U_L a constant). For time $t > 0$, its ends $x = 0$ and $x = L$ are held at temperatures 0°C and U_L°C, respectively. Find the temperature distribution in the rod for $t > 0$.

2. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at constant temperature U_0°C throughout. For $t > 0$, heat is added at the end $x = 0$ at a constant rate Q W/m^2, and end $x = L$ continues to be held at temperature U_0°C. Find the temperature in the rod for $0 < x < L$ and $t > 0$.

3. A cylindrical, homogeneous, isotropic rod with insulated sides is initially at temperature $U_0(1 - x/L)$, where U_0 is a constant. For time $t > 0$, the end $x = 0$ is maintained at temperature U_0 and end $x = L$ is insulated. Find the temperature in the rod for $0 < x < L$ and $t > 0$.

4. Repeat Exercise 3 if heat is added uniformly over the end $x = L$ at a constant rate Q W/m^2.

5. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature 20°C throughout ($0 \le x \le L$) at time $t = 0$. For $t > 0$, a constant electric current I is passed along the length of the rod, creating heat generation $g(x, t) = I^2/(A^2\sigma)$, where σ is the electrical conductivity of the rod and A is its cross-sectional area. If the ends of the rod are held at temperature 0°C for $t > 0$, find the temperature in the rod for $t > 0$ and $0 < x < L$.

6. Repeat Exercise 5 if the ends of the rod are held at temperature 100°C for $t > 0$.

7. Repeat Exercise 5 if the ends $x = 0$ and $x = L$ of the rod are held at temperatures U_0 and U_L, respectively, for $t > 0$.

Part B Vibration Problems

8. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial displacement at time $t = 0$ of $f(x)$, $0 \le x \le L$, and an initial velocity $g(x)$, $0 \le x \le L$. If an external force per unit length of constant magnitude acts vertically downward at every point on the string, find a series representation for displacements in the string for $t > 0$ and $0 < x < L$.

9. A taut string has an end at $x = 0$ fixed on the x-axis, but the end at $x = L$ is removed a small amount y_L away from the x-axis and kept at this position. If the string has initial position $f(x)$ and velocity $g(x)$ (at time $t = 0$), find a series representation for displacements for $t > 0$ and $0 < x < L$.

10. The end $x = 0$ of a taut string is fixed on the x-axis. The end $x = L$ is looped around a smooth vertical support. If the string falls from rest along the x-axis, and a constant vertical force F_0 acts on the loop at $x = L$, find displacements of the string. Take gravity into account.

Part C Potential Problems

11. Find a formula for the solution of Laplace's equation inside the rectangle $0 \le x \le L$, $0 \le y \le L'$ in Figure 21.14 when $g_1(y) = h_1(x) = 0$ and $g_2(y) = h_2(x) = 1$.

ANSWERS

1. $U_L x/L$

2. $\dfrac{Q}{\kappa}(L-x) + U_0 - \dfrac{8QL}{\kappa\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2kt/(4L^2)}\cos\frac{(2n-1)\pi x}{2L}$

3. $U_0 + \dfrac{8U_0}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}e^{-(2n-1)^2\pi^2kt/(4L^2)}\sin\frac{(2n-1)\pi x}{2L}$

4. $\dfrac{Qx}{\kappa} + U_0 + \dfrac{8(U_0\kappa + QL)}{\kappa\pi^2}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}e^{-(2n-1)^2\pi^2kt/(4L^2)}\sin\frac{(2n-1)\pi x}{2L}$

5. $\dfrac{I^2x(L-x)}{2\kappa A^2\sigma} + \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{20}{2n-1} - \frac{I^2L^2}{\kappa A^2\sigma\pi^2(2n-1)^3}\right]e^{-(2n-1)^2\pi^2kt/L^2}\sin\frac{(2n-1)\pi x}{L}$

6. $100 + \dfrac{I^2x(L-x)}{2\kappa A^2\sigma} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\left[\frac{80}{2n-1} + \frac{I^2L^2}{\kappa A^2\sigma\pi^2(2n-1)^3}\right]e^{-(2n-1)^2\pi^2kt/L^2}\sin\frac{(2n-1)\pi x}{L}$

7. $\dfrac{I^2x(L-x)}{2\kappa A^2\sigma} + \left(\dfrac{U_L - U_0}{L}\right)x + U_0 - \dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{U_0 + U_L(-1)^{n+1}}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}$

$$+\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{20}{2n-1} - \frac{I^2L^2}{\kappa A^2\sigma\pi^2(2n-1)^3}\right]e^{-(2n-1)^2\pi^2kt/L^2}\sin\frac{(2n-1)\pi x}{L}$$

8. $-\dfrac{kx(L-x)}{2\rho c^2} + \displaystyle\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$, where

$$a_n = \frac{2}{L}\int_0^L\left[f(x) + \frac{kx(L-x)}{2\rho c^2}\right]\sin\frac{n\pi x}{L}dx, \quad b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

9. $\psi(x) + \displaystyle\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L} + b_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$, where

$$\psi(x) = \frac{y_Lx}{L} \quad a_n = \frac{2}{L}\int_0^L[f(x) - \psi(x)]\sin\frac{n\pi x}{L}dx, \quad b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

10. $-\dfrac{gx(x-2L)}{2c^2} + \dfrac{F_0x}{\tau} + \dfrac{8L}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\left[\frac{2Lg}{(2n-1)^3\pi c^2} + \frac{F_0(-1)^{n+1}}{(2n-1)^2\tau}\right]\cos\frac{(2n-1)\pi ct}{2L}\sin\frac{(2n-1)\pi x}{2L}$

11. $$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi y/L}-e^{-(2n-1)\pi y/L}}{e^{n\pi L'/L}-e^{-n\pi L'/L}}\sin\frac{(2n-1)\pi x}{L}$$
$$+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi x/L'}-e^{-(2n-1)\pi x/L'}}{e^{n\pi L/L'}-e^{-n\pi L/L'}}\sin\frac{(2n-1)\pi y}{L'}$$