UNIVERSITY OF MANITOBA

DATE: February 16, 2011 — MIDTERM I

TITLE PAGE

COURSE: MATH 3132 — TIME: 60 minutes

EXAMINATION: Engineering Mathematical Analysis 3 — EXAMINER: M. Davidson

---

FAMILY NAME: (Print in ink) ____________________

GIVEN NAME(S): (Print in ink) ____________________

STUDENT NUMBER: ____________

SIGNATURE: (in ink) ____________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 60 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 5 pages of questions, the last of which contains formulas. Please check that you have all the pages.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 40 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	8	
2	9	
3	11	
4	12	
Total:	40	

---

[8] 1. Evaluate the following line integral:

$$\int_C xy^2 + yz^2 \ \mathrm{ds}$$

where C is the curve $x = \sin^3 t, \quad y = 2, \quad z = \cos^3 t, \quad \text{for } 0 \le t \le \frac{\pi}{2}.$

Solution:

$$\int_C x^2y + yz^2 \ \mathrm{ds}$$

$$= \int_0^{\frac{\pi}{2}} (\sin^3 t)(2)^2 + (2)(\cos^3 t)^2 \sqrt{(3\sin^2 t \cos t)^2 + (0)^2 + (3\cos^2 t \sin t)^2} \ \mathrm{dt}$$

$$= \int_0^{\frac{\pi}{2}} (4\sin^3 t + 2\cos^6 t) \sqrt{9\sin^4 t \cos^2 t + 9\cos^4 t \sin^2 t} \ \mathrm{dt}$$

$$= \int_0^{\frac{\pi}{2}} (4\sin^3 t + 2\cos^6 t) \sqrt{9\sin^2 t \cos^2 t(\sin^2 t + \cos^2 t)} \ \mathrm{dt}$$

$$= \int_0^{\frac{\pi}{2}} (4\sin^3 t + 2\cos^6 t)(3\sin t \cos t) \ \mathrm{dt}$$

$$= \int_0^{\frac{\pi}{2}} (12\sin^4 t \cos t + 6\cos^7 t \sin t) \ \mathrm{dt}$$

$$= \left(\frac{12}{5}\sin^5 t\right)_0^{\frac{\pi}{2}} + \left(\frac{-6}{8}\cos^8 t\right)_0^{\frac{\pi}{2}}$$

$$= \left(\frac{12}{5}\right) + \left(\frac{6}{8}\right) = \frac{63}{20}$$

---

[9] 2. (a) For what region(s) is $\int_C \vec{F}\cdot d\vec{r}$ is independent of path where

$$\vec{F} = \left(\frac{2x}{y}\right)\hat{\imath} + \left(\frac{-(x^2+z^2)}{y^2}\right)\hat{\jmath} + \left(\frac{2z}{y}\right)\hat{k}\,.$$

Solution: For the function $\phi = \dfrac{x^2+z^2}{y}$, we get that $\nabla\phi = \vec{F}$, hence $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in any region which does not contain points where $y = 0$.
Note: This can also be done by showing that $\nabla\times\vec{F} = \vec{0}$

(b) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}$ is given in part(a) and C is the curve
$x = t^2 - 5t + 3$,
$y = t$,
$z = t^3 - 7t^2 + 13t - 9$, for $1 \le t \le 5$.

Solution: At $t =$ we have the point $(-1, 1, -2)$.
At $t = 5$ we have the point $(3, 5, 6)$.
Since $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in a region containing the given curve we get

$$\begin{aligned}
&\int_C \vec{F}\cdot d\vec{r}\\
&= \left(\frac{x^2+z^2}{y}\right)_{(-1,1,-2)}^{(3,5,6)}\\
&= \left(\frac{(3)^2+(6)^2}{5}\right) - \left(\frac{(-1)^{+}(-2)^2}{1}\right)\\
&= 9 - 5 = 4
\end{aligned}$$

---

[11] 3. Evaluate the closed line integral $\oint_C \vec{F} \cdot \mathrm{d}\vec{r}$ where

C is the piecewise smooth curve formed by the curve $y = 8 - 2x^2$ from $x = -2$ to $x = 2$ and the x-axis from $x = 2$ to $x = -2$ and

$$\vec{F} = (x^2 \sin x^4 + 4y\cos(xy) + 9x^2y^2 + 3y - 2xy)\,\hat{i} + (y^4 e^{y^2} + 4x\cos(xy) + 6x^3y + 2x + x^2)\,\hat{j}.$$

Solution:

$$\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y$$

$$= -\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y$$

Applying Greens Theorem:

$$= -\iint_R (4\cos xy - 4xy\sin xy + 18x^2y + 2 + 2x) - (4\cos xy - 4xy\sin xy + 18x^2y + 3 - 2x)\mathrm{dA}$$

$$= -\int_{-2}^{2}\int_0^{8-2x^2} 4x - 1\,\mathrm{d}y\,\mathrm{d}x$$

$$= -\int_{-2}^{2} [(4x-1)y]_0^{8-2x^2}\,\mathrm{d}x$$

$$= -\int_{-2}^{2} (4x-1)(8-2x^2)\,\mathrm{d}x$$

$$= -\int_{-2}^{2} 32x - 8x^3 - 8 + 2x^2\,\mathrm{d}x$$

$$= -\,[16x^2 - 2x^4 - 8x + \frac{2}{3}x^3]_{-2}^{2}$$

$$= -\,(-16 + \frac{16}{3} - 16 + \frac{16}{3}) = \frac{64}{3}$$

---

[12] 4. Evaluate the surface integral $\iint_S \vec{F} \cdot \hat{n}\, \mathrm{d}S$ where

$$\vec{F} = (2x^3 - 3xy^2 + yz^2)\,\hat{\imath} + (2y^3 - 3x^2y + x^2z)\,\hat{\jmath} + (z+1)\hat{k}$$

and S is the portion of the paraboloid $z = 16 - 4x^2 - 4y^2$ which lies above the xy-plane and $\hat{n}$ is the upward normal to that surface.

HINT: THIS MAY BE EASIER WITH THE APPROPRIATE APPLICATION OF A PARTICULAR THEORM

Solution:

We use the Divergence theorem on the closed surface $S + S_1$ where S_1 is the surface $z = 0$ in the circle $x^2 + y^2 = 4$ (later denoted D) with $\hat{n} = -\hat{k}$.
We get:

$$\oiint_{S+S_1} \vec{F} \cdot \hat{n}\ \mathrm{d}S$$

$$= \iiint_V \nabla \cdot \vec{F}\ \mathrm{d}V$$

$$= \iiint_V (3x^2 + 3y^2 + 1)\ \mathrm{d}V$$

Using cylindrical coordinates we get:

$$= \int_0^2 \int_0^{2\pi} \int_0^{16-4r^2} (3r^2 + 1)r\ \ \mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}r$$

$$= \int_0^2 \int_0^{2\pi} \Big[(3r^3 + r)z\Big]_0^{16-4r^2}\ \mathrm{d}\theta\,\mathrm{d}r$$

$$= \int_0^2 \int_0^{2\pi} (3r^3 + r)(16 - 4r^2)\ \mathrm{d}\theta\,\mathrm{d}r$$

$$= \int_0^2 \int_0^{2\pi} (44r^3 - 12r^5 + 16r)\ \ \mathrm{d}\theta\,\mathrm{d}r$$

$$= \int_0^2 \Big[(44r^3 - 12r^5 + 16r)\theta\Big]_0^{2\pi}\ \ \mathrm{d}r$$

$$=2\pi \int_0^2 (44r^3 - 12r^5 + 16r)\ \mathrm{d}r$$

$$=2\pi \Big[11r^4 - 2r^6 + 8r^2\Big]_0^2\ \ \mathrm{d}r$$

$$=2\pi(80) = 160\pi$$

So $\iint_S \vec{F} \cdot \hat{n}\,\mathrm{d}S = \oiint_{S+S_1} \vec{F} \cdot \hat{n}\ \mathrm{d}S - \iint_{S_1} \vec{F} \cdot \hat{n}\,\mathrm{d}S$

$\iint_{S_1} \vec{F} \cdot \hat{n}\,\mathrm{d}S = \iint_{S_1} (-z - 1)\mathrm{d}S = \iint_D (-1)\mathrm{d}A = -\iint 1\mathrm{d}A = -1(\text{area}) = -4\pi$

Finally $\iint_S \vec{F} \cdot \hat{n}\,\mathrm{d}S = 160\pi - (-4\pi) = 164\pi$.