UNIVERSITY OF MANITOBA

DATE: March 23, 2011 | MIDTERM II

TITLE PAGE

COURSE: MATH 3132 | TIME: 60 minutes

EXAMINATION: Engineering Mathematical Analysis 3 | EXAMINER: M. Davidson

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FAMILY NAME: (Print in ink) ____________________

GIVEN NAME(S): (Print in ink) ____________________

STUDENT NUMBER: __________

SIGNATURE: (in ink) ____________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 60 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 8 pages of questions, the last of which contains formulas. Please check that you have all the pages.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 50 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	10	
2	10	
3	8	
4	12	
5	10	
Total:	50	

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[10] 1. Use Stokes's Theorem to evaluate $\oint_C \vec{F} \cdot \mathrm{d}\vec{r}$ where

$$\vec{F} = (xyz)\hat{\imath} + (y)\hat{\jmath} + (z)\hat{k}\,.$$

and C is the curve of intersection of $z = xy$ and $x^2 + y^2 = 4$ oriented clockwise when viewed from above.

Solution:

We will use Stokes theorem for this question: $\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,\mathrm{d}S$

Finding the curl of $\vec{F}$:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & z \end{vmatrix} = (0-0)\hat{\imath} - (0-xy)\hat{\jmath} + (0-xz)\hat{k} = xy\hat{\jmath} - xz\hat{k}\,.$$

We take as the surface S, $z = xy$ where S_{xy} is the inside of the circle $x^2 + y^2 = 4$; $\hat{n}$ is downward so using $\nabla(xy - z)$ we find $\hat{n} = \frac{1}{\sqrt{x^2+y^2+1}}(y\hat{\imath} + x\hat{\jmath} - \hat{k})$ and $\mathrm{d}S = \sqrt{x^2+y^2+1}\,\mathrm{d}A$.

$$\begin{aligned}
\oint_C \vec{F} \cdot \mathrm{d}\vec{r} &= \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,\mathrm{d}S \\
&= \iint_S (xy\hat{\jmath} - xz\hat{k}) \cdot \frac{1}{\sqrt{x^2+y^2+1}}(y\hat{\imath} + x\hat{\jmath} - \hat{k})\,\mathrm{d}s \\
&= \iint_{S_{xy}} (x^2y + xz)\frac{1}{\sqrt{x^2+y^2+1}}\,\sqrt{x^2+y^2+1}\,\mathrm{d}A \\
&= \iint_{S_{xy}} x^2y + x^2y\,\mathrm{d}A \\
&= 2\int_0^{2\pi}\int_0^2 r^4\cos^2\theta\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta \\
&= 2\int_0^{2\pi} \frac{1}{5}\,r^5\cos^2\theta\sin\theta\Big]_0^2\,\mathrm{d}\theta \\
&= \frac{64}{5}\int_0^{2\pi}\cos^2\theta\sin\theta\,\mathrm{d}\theta \\
&= \frac{64}{5}\left[-\frac{1}{3}\cos^3\theta\right]_0^{2\pi} = 0
\end{aligned}$$

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[4] 2. (a) Find k (a constant) so that the functions $f(x) = x^2 + kx$ and $g(x) = 20x$ are orthogonal with respect to the weight function $w(x) = x$ on the interval $0 \le x \le 1$.

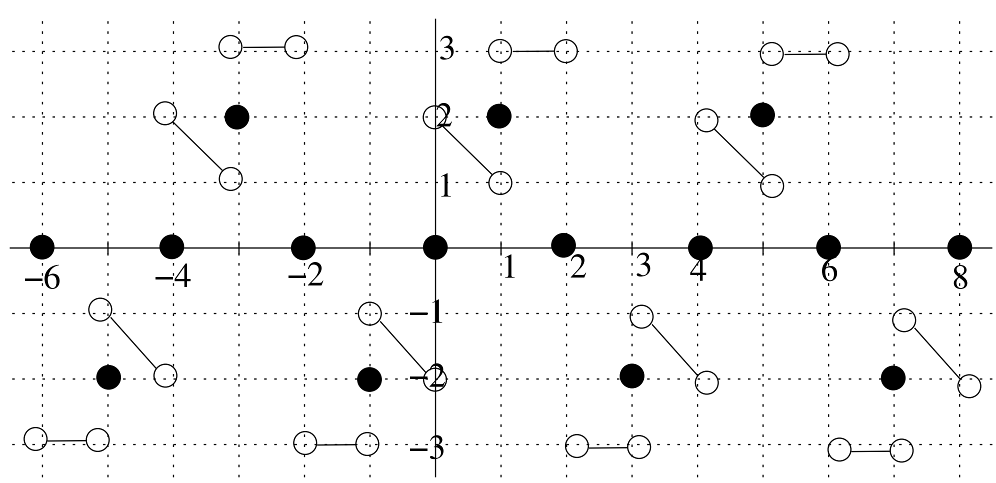
Solution:

$$\begin{aligned}\int_0^1 w(x)\, f(x)\, g(x)\, \mathrm{d}x &= \int_0^1 (x)\,(x^2 + kx)\,(20x)\, \mathrm{d}x \\ &= \left[\frac{20}{5}x^5 + \frac{20}{4}kx^4\right]_0^1 \\ &= 4 + 5k\end{aligned}$$

For orthogonal, we want $4 + 5k = 0$ so $k = \frac{-4}{5}$

[3] (b) Draw the graph of the Fourier Sine series of the function $f(x) = \begin{cases} 2 - x & 0 < x \le 1 \\ 3 & 1 < x \le 2 \end{cases}$.

(Do not solve for any coefficients)



[3] (c) Find the singular points of $(x^2 - 9)y'' + xy' + y = 0$. If a solution to this equation is of the form $y = \sum_{n=0}^{\infty} c_n\,(x-1)^n$, what is the smallest the radius of convergence will be? (Why?)

Solution:
Since $P(x) = x^2 - 9$, $Q(x) = x$ and $R(x) = 1$, the functions $\frac{x}{x^2-9}$ and $\frac{1}{x^2-9}$ do not have Taylor series expansions about $c = \pm 3$. So $c = \pm 3$ are the singular points.
The smallest radius of convergence of the solution about the point 1 (which is an ordinary point) would be 2, which is the distance to the nearest singular point.

[8] 3. Find the Fourier Cosine Series expansion of $f(x) = 3x$ on $0 \le x \le 2$

Solution:

We have (since $L = 2$)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{2} x$$

Solving for a_0:

$$\begin{aligned} a_0 &= \frac{2}{2} \int_0^2 3x \, \mathrm{d}x \\ &= \frac{3}{2} x^2 \Big]_0^2 = 6\,. \end{aligned}$$

Solving for a_n where $n \ge 1$:

$$\begin{aligned} a_n &= \frac{2}{2} \int_0^2 3x \cos \frac{n\pi}{2} x \, \mathrm{d}x \\ &= 3 \left[\frac{4}{n^2\pi^2} \cos \frac{n\pi}{2} x + \frac{2x}{n\pi} \sin \frac{n\pi}{2} x \right]_0^2 \\ &= 3 \left[\frac{4}{n^2\pi^2} \cos n\pi + \frac{4}{n\pi} \sin n\pi - \frac{4}{n^2\pi^2} - 0 \right] \\ &= \frac{12}{n^2\pi^2} ((-1)^n - 1)\,. \end{aligned}$$

So we have

$$f(x) = 3 + \sum_{n=1}^{\infty} \frac{12}{n^2\pi^2} ((-1)^n - 1) \cos \frac{n\pi}{2} x\,.$$

(If we wish to eliminate the even terms, which are all zero, we get:

$$f(x) = 3 + \sum_{n=1}^{\infty} \frac{-24}{(2n-1)^2\pi^2} \cos \frac{(2n-1)\pi}{2} x \Bigg).$$

[6] 4. (a) Consider the differential equation:

$$(x^2+2)y'' - y = 0$$

A solution of the above differental equation can be represented by :

$$y = \sum_{n=0}^{\infty} c_n\, x^n.$$

Find a recurrence relation for the c_n and simplify it as much as possible.
Do not attempt to iterate this recurrence relation, do not solve for the c_n or calculate any solution for $y(x)$.

Solution:

Since $y = \sum_{n=0}^{\infty} c_n\, x^n$, we have $y' = \sum_{n=0}^{\infty} nc_n\, x^{n-1}$, and $y'' = \sum_{n=0}^{\infty} n(n-1)c_n\, x^{n-2}$.

So

$$\begin{aligned}
(x^2+2)y'' - y &= (x^2+2)\sum_{n=0}^{\infty} n(n-1)c_n\, x^{n-2} - \sum_{n=0}^{\infty} c_n\, x^n \\
&= \sum_{n=0}^{\infty} n(n-1)c_n\, x^n + \sum_{n=0}^{\infty} 2n(n-1)c_n\, x^{n-2} - \sum_{n=0}^{\infty} c_n\, x^n \\
&= \sum_{n=0}^{\infty} n(n-1)c_n\, x^n + \sum_{n=0}^{\infty} 2(n+2)(n+1)c_{n+2}\, x^n - \sum_{n=0}^{\infty} c_n\, x^n \\
&= \sum_{n=0}^{\infty} (n(n-1)c_n + 2(n+2)(n+1)c_{n+2} - c_n)\, x^n \\
&= \sum_{n=0}^{\infty} ((n^2-n-1)c_n + 2(n+2)(n+1)c_{n+2})\, x^n
\end{aligned}$$

Since $(x^2+2)y'' - y = 0$ we have that $(n^2-n-1)c_n + 2(n+2)(n+1)c_{n+2} = 0$.

So $c_{n+2} = \dfrac{-(n^2-n-1)c_n}{2(n+2)(n+1)}$.

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[6] (b) When a solution of the differential equation:

$$(x^2+1)y'' - 6y = 0$$

is represented by : $y = \sum_{n=0}^{\infty} c_n x^n$, its coefficients, c_n, satisfy the recurrence relation: $c_{n+2} = -\dfrac{(n-3)}{(n+1)} c_n$ (Do not show this, you may assume the above recurrence relation.)

Use the above recurrence relation to find two linearly independent solutions of the above differential equation. Write any infinite series using sigma notation.

Solution: Since the recurrence relation is of order 2, we see that we will get our independent solutions from c_0 and c_1.

Finding a few terms for c_{2n}:

$n=0 \quad c_2 = -\frac{-3}{1}c_0 = 3c_0$

$n=2 \quad c_4 = -\frac{-1}{3}c_2 = c_0$

$n=4 \quad c_6 = -\frac{1}{5}c_4 = -\frac{1}{5}c_0$

$n=6 \quad c_8 = -\frac{3}{7}c_6 = (-1)^2\frac{3\cdot 1}{7\cdot 5}c_0$

$n=8 \quad c_{10} = -\frac{5}{9}c_8 = (-1)^3\frac{5\cdot 3\cdot 1}{9\cdot 7\cdot 5}c_0$

$n=10 \quad c_{12} = -\frac{7}{11}c_{10} = (-1)^4\frac{7\cdot 5\cdot 3\cdot 1}{11\cdot 9\cdot 7\cdot 5}c_0$

We now recognize a pattern which is (for at least $n \geq 4$):

$$c_{2n} = (-1)^n \frac{(2n-5)(2n-7)\cdots(-1)(-3)}{(2n-1)(2n-3)\cdots(5)(3)} c_0 = (-1)^n \frac{3}{(2n-1)(2n-3)} c_0$$

So one solution is:

$$y_0(x) = c_0 \left[1 + 3x^2 + x^4 - \frac{1}{5}x^6 + \sum_{n=4}^{\infty} (-1)^n \frac{3}{(2n-1)(2n-3)} x^{2n} \right]$$

(This can be simplified to

$$y_0(x) = c_0 \left[\sum_{n=0}^{\infty} (-1)^n \frac{3}{(2n-1)(2n-3)} x^{2n} \right]$$

if you check the leading terms.)

For terms c_{2n-1}:

$n=1 \quad c_3 = -\frac{-2}{2}c_1 = c_1$

$n=3 \quad c_5 = -\frac{0}{4}c_3 = 0$

$n=5 \quad c_7 = -\frac{2}{6}c_5 = 0$

The remaining odd terms will all be zero.

The other solution is

$$y_1(x) = c_1(x + x^3).$$

[10] 5. Solve the eigenvalue problem $y'' + \lambda y = 0$ with boundary conditions $y'(0) = 0$ and $y(2) = 0$. You may assume FOR THIS QUESTION that $\lambda > 0$.

[Bonus:]
Show that eigenfunctions from distinct eigenvalues for the above problem are orthogonal on $0 \le x \le 2$ with respect to the weight function $w(x) = 1$.

Solution: Since $\lambda > 0$ we let $\lambda = \alpha^2$ where $\alpha > 0$.

Hence $y(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$.
and $y'(x) = -\alpha c_1 \sin \alpha x + \alpha c_2 \cos \alpha x$.

Considering $y'(0) = 0$, we get $\alpha c_2 = 0$ and since $\alpha \neq 0$ we get $c_2 = 0$.

Now our function is $y(x) = c_1 \cos \alpha x$ and we want $c_1 \neq 0$, else our function would be the zero function.
From $y(2) = 0$ we have $c_1 \cos 2\alpha = 0$, since $c_1 \neq 0$ we get that
$\cos 2\alpha = 0$ hence
$2\alpha = \dfrac{(2n-1)\pi}{2}$ for $n \geq 1$.
So $\alpha = \dfrac{(2n-1)\pi}{4}$.

Hence our eigenfunctions are $\lambda_n = \dfrac{(2n-1)^2\pi^2}{16}$
with associated eigenfunctions $y_n(x) = \cos \dfrac{(2n-1)\pi}{4} x$.

Some formulas from Vector Calculus:

Green's Theorem:

$$\oint_C P\,\mathrm{dx} + Q\,\mathrm{dy} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A$$

Divergence Theorem:

$$\oiint_S \vec{F} \cdot \hat{n}\ \mathrm{d}S = \iiint_V \nabla \cdot \vec{F}\ \mathrm{d}V$$

Stokes's Theorem:

$$\oint_C \vec{F} \cdot \mathrm{d}\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,\mathrm{d}S$$

The following are true for any interger n:

$$\sin(n\pi) = 0, \qquad \cos(n\pi) = (-1)^n, \qquad \sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n, \qquad \cos\left(\frac{(2n+1)\pi}{2}\right) = 0.$$

Some trigonometric formulas:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

$$\cos A \cos B = \frac{1}{2}(\cos(A+B) + \cos(A-B))$$

$$\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

$$\cos^2 A = \frac{1}{2} + \frac{1}{2}\cos 2A$$

$$\sin^2 A = \frac{1}{2} - \frac{1}{2}\cos 2A$$

(Full) Fourier Series:

Let $f(x)$ be defined and piece-wise smooth on $\{x : 0 \leq x \leq 2L\}$. The **(Full) Fourier series** of $f(x)$ is given by :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Half-range expansions:

Let $f(x)$ be defined and piece-wise smooth on $\{x : 0 \leq x \leq L\}$.

The **Fourier Cosine series** of $f(x)$ is given by :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

$$a_n = \frac{2}{L}\int_0^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

The **Fourier Sine series** of $f(x)$ is given by :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x$$

$$b_n = \frac{2}{L}\int_0^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

Some trigonometric integrals:

You may use any of the following formulas without further explanation:
(k is a non-zero constant)

$$\int \cos kx\,dx = \frac{1}{k}\sin kx \qquad \int \sin kx\,dx = -\frac{1}{k}\cos kx$$

$$\int x\cos kx\,dx = \frac{1}{k^2}\cos kx + \frac{x}{k}\sin kx \qquad \int x\sin kx\,dx = \frac{1}{k^2}\sin kx - \frac{x}{k}\cos kx$$

$$\int x^2\cos kx\,dx = \frac{x^2}{k}\sin kx + \frac{2x}{k^2}\cos kx - \frac{2}{k^3}\sin kx$$

$$\int x^2\sin kx\,dx = -\frac{x^2}{k}\cos kx + \frac{2x}{k^2}\sin kx + \frac{2}{k^3}\cos kx$$