EXERCISES 21.3

2. The initial boundary value problem for temperature in the rod is

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0,\\
U_x(0,t) &= -Q/\kappa, \quad t > 0,\\
U(L,t) &= U_0, \quad t > 0,\\
U(x,0) &= U_0, \quad 0 < x < L.
\end{aligned}$$

We define a new dependent variable $V(x,t)$ by $U(x,t) = V(x,t) + \psi(x)$ where $\psi(x)$ is the solution of the associated steady-state problem

$$\begin{aligned}
k\frac{d^2\psi}{dx^2} &= 0, \quad 0 < x < L,\\
\psi'(0) &= -Q/\kappa,\\
\psi(L) &= U_0.
\end{aligned}$$

The differential equation implies that $\psi(x) = Ax + B$, and the boundary conditions require

$$-Q/\kappa = A, \qquad U_0 = AL + B.$$

From these, we obtain the steady-state solution

$$\psi(x) = \frac{Q}{\kappa}(L - x) + U_0.$$

With this choice for $\psi(x)$, the PDE for $V(x,t)$ can be found by substituting $U(x,t) = V(x, t + \psi(x)$ into the PDE for $U(x,t)$,

$$\frac{\partial}{\partial t}[V(x,t) + \psi(x)] = k\frac{\partial^2}{\partial x^2}[V(x,t) + \psi(x)].$$

Because $\psi(x)$ is only a function of x that has a vanishing second derivative, this equation simplifies to

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0.$$

Boundary conditions for $V(x,t)$ are obtained from representation $U(x,t) = V(x,t) + \psi(x)$ and boundary conditions $U(x,t)$:

$$\begin{aligned}
V_x(0,t) &= U_x(0,t) - \psi'(0) = -Q/\kappa + Q/\kappa = 0, \quad t > 0,\\
V(L,t) &= U(L,t) - \psi(L) = U_0 - U_0 = 0, \quad t > 0.
\end{aligned}$$

Finally, $V(x,t)$ must satisfy the initial condition

$$V(x,0) = U(x,0) - \psi(x) = U_0 - \frac{Q}{\kappa}(L - x) - U_0 = -\frac{Q}{\kappa}(L - x), \quad 0 < x < L.$$

Separation of variables $V(x,t) = X(x)T(t)$ on the PDE and boundary conditions leads to the ordinary differential equations

$$\begin{aligned}
X'' + \lambda X &= 0, \quad 0 < x < L, \qquad\qquad T' + k\lambda T = 0, \quad t > 0.\\
X'(0) &= X(L) = 0;
\end{aligned}$$

The Sturm-Liouville system was discussed in Section 19.2. According to line 4 of Table 19.1, eigenvalues are $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$ and corresponding eigenfunctions are $X_n(x) = \cos\dfrac{(2n-1)\pi x}{2L}$. Since the auxiliary equation for the differential equation in $T(t)$ is $m + k\lambda_n = 0$, with solution $m = -k\lambda_n$, a general solution of the differential equation is $T(t) = be^{-k\lambda_n t} = be^{-(2n-1)^2\pi^2 kt/(4L^2)}$. Separated functions are $be^{-(2n-1)^2\pi^2 kt/(4L^2)} \cos \frac{(2n-1)\pi x}{2L}$. Because the PDE and boundary conditions are linear and homogeneous, we superpose separated functions in the form

$$V(x,t) = \sum_{n=1}^{\infty} b_n e^{-(2n-1)^2\pi^2 kt/(4L^2)} \cos \frac{(2n-1)\pi x}{2L}.$$

The initial condition on $V(x,t)$ requires the constants b_n to satisfy

$$-\frac{Q}{\kappa}(L-x) = \sum_{n=1}^{\infty} b_n \cos \frac{(2n-1)\pi x}{2L}, \quad 0 < x < L.$$

Consequently, the b_n are the coefficients in the eigenfunction expansion of the the function on the left in terms of the eighenfunctions on the right. Thus,

$$b_n = \frac{2}{L} \int_0^L -\frac{Q}{\kappa}(L-x) \cos \frac{(2n-1)\pi x}{2L} dx.$$

Integration by parts leads to

$$b_n = -\frac{8QL}{(2n-1)^2\pi^2\kappa}.$$

The formal solution of the problem is therefore

$$\begin{aligned} U(x,t) &= \frac{Q}{\kappa}(L-x) + U_0 + V(x,t) \\ &= \frac{Q}{\kappa}(L-x) + U_0 + \sum_{n=1}^{\infty} \frac{-8QL}{(2n-1)^2\pi^2\kappa} e^{-(2n-1)^2\pi^2 kt/(4L^2)} \cos \frac{(2n-1)\pi x}{2L} \\ &= \frac{Q}{\kappa}(L-x) + U_0 - \frac{8QL}{\kappa\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2\pi^2 kt/(4L^2)} \cos \frac{(2n-1)\pi x}{2L}. \end{aligned}$$

4. The initial boundary value problem for temperature in the rod is

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \\ U(0,t) &= U_0, \quad t > 0, \\ U_x(L,t) &= Q/\kappa, \quad t > 0, \\ U(x,0) &= U_0(1 - x/L), \quad 0 < x < L. \end{aligned}$$

We define a new dependent variable $V(x,t)$ by $U(x,t) = V(x,t) + \psi(x)$ where $\psi(x)$ is the solution of the associated steady-state problem

$$\begin{aligned} k\frac{d^2\psi}{dx^2} &= 0, \quad 0 < x < L, \\ \psi(0) &= U_0, \\ \psi'(L) &= Q/\kappa. \end{aligned}$$

The differential equation implies that $\psi(x) = Ax + B$, and the boundary conditions require

$$U_0 = B, \qquad Q/\kappa = A.$$

From these, we obtain the steady-state solution

$$\psi(x) = \frac{Qx}{\kappa} + U_0.$$

With this choice for $\psi(x)$, the PDE for $V(x,t)$ can be found by substituting $U(x,t) = V(x,t+\psi(x)$ into the PDE for $U(x,t)$,

$$\frac{\partial}{\partial t}[V(x,t)+\psi(x)] = k\frac{\partial^2}{\partial x^2}[V(x,t)+\psi(x)].$$

Because $\psi(x)$ is only a function of x that has a vanishing second derivative, this equation simplifies to

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0.$$

Boundary conditions for $V(x,t)$ are obtained from representation $U(x,t) = V(x,t)+\psi(x)$ and boundary conditions $U(x,t)$:

$$V(0,t) = U(0,t) - \psi(0) = U_0 - U_0 = 0, \quad t > 0,$$
$$V_x(L,t) = U_x(L,t) - \psi'(L) = Q/\kappa - Q/\kappa = 0, \quad t > 0.$$

Finally, $V(x,t)$ must satisfy the initial condition

$$V(x,0) = U(x,0) - \psi(x) = U_0\left(1-\frac{x}{L}\right) - \frac{Qx}{\kappa} - U_0, \quad 0 < x < L.$$

Separation of variables $V(x,t) = X(x)T(t)$ on the PDE and boundary conditions leads to the ordinary differential equations

$$X'' + \lambda X = 0, \quad 0 < x < L, \qquad T' + k\lambda T = 0, \quad t > 0.$$
$$X(0) = X'(L) = 0;$$

The Sturm-Liouville system was discussed in Section 19.2. According to line 3 of Table 19.1, eigenvalues are $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$ and corresponding eigenfunctions are $X_n(x) = \sin\dfrac{(2n-1)\pi x}{2L}$. Since the auxiliary equation for the differential equation in $T(t)$ is $m + k\lambda_n = 0$, with solution $m = -k\lambda_n$, a general solution of the differential equation is $T(t) = be^{-k\lambda_n t} = be^{-(2n-1)^2\pi^2 kt/(4L^2)}$. Separated functions are $be^{-(2n-1)^2\pi^2 kt/(4L^2)}\sin\dfrac{(2n-1)\pi x}{2L}$. Because the PDE and boundary conditions are linear and homogeneous, we superpose separated functions in the form

$$V(x,t) = \sum_{n=1}^{\infty} b_n e^{-(2n-1)^2\pi^2 kt/(4L^2)}\sin\frac{(2n-1)\pi x}{2L}.$$

The initial condition on $V(x,t)$ requires the constants b_n to satisfy

$$U_0\left(1-\frac{x}{L}\right) - \frac{Qx}{\kappa} - U_0 = \sum_{n=1}^{\infty} b_n \sin\frac{(2n-1)\pi x}{2L}, \quad 0 < x < L.$$

Consequently, the b_n are the coefficients in the eigenfunction expansion of the the function on the left in terms of the eighenfunctions on the right. Thus,

$$b_n = \frac{2}{L}\int_0^L \left[U_0\left(1-\frac{x}{L}\right) - \frac{Qx}{\kappa} - U_0\right]\sin\frac{(2n-1)\pi x}{2L}dx.$$

Integration by parts leads to

$$b_n = \frac{8L(-1)^n}{(2n-1)^2\pi^2}\left(\frac{U_0}{L} + \frac{Q}{\kappa}\right).$$

The formal solution of problem is therefore

$$U(x,t) = \frac{Qx}{\kappa} + U_0 + V(x,t)$$

$$= \frac{Qx}{\kappa} + U_0 + \sum_{n=1}^{\infty} \frac{8L(-1)^n}{(2n-1)^2\pi^2} \left(\frac{U_0}{L} + \frac{Q}{\kappa} \right) e^{-(2n-1)^2\pi^2 kt/(4L^2)} \sin \frac{(2n-1)\pi x}{2L}$$

$$= \frac{Qx}{\kappa} + U_0 + \frac{8(U_0\kappa + QL)}{\kappa\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} e^{-(2n-1)^2\pi^2 kt/(4L^2)} \sin \frac{(2n-1)\pi x}{2L}.$$

6. The initial boundary value problem for $U(x,t)$ is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{kI^2}{\kappa A^2 \sigma}, \quad 0 < x < L, \quad t > 0,$$
$$U(0,t) = 100, \quad t > 0,$$
$$U(L,t) = 100, \quad t > 0,$$
$$U(x,0) = 20, \quad 0 < x < L.$$

Because the nonhomogeneities are time-independent, we set $U(x,t) = V(x,t) + \psi(x)$, where $\psi(x)$ is the steady-state solution satisfying

$$k\frac{d^2\psi}{dx^2} + \frac{kI^2}{\kappa A^2\sigma} = 0, \quad 0 < x < L,$$
$$\psi(0) = \psi(L) = 100.$$

Integration of the differential equation gives

$$\psi(x) = -\frac{I^2 x^2}{2\kappa A^2 \sigma} + Ax + B.$$

The boundary conditions require

$$100 = \psi(0) = B, \qquad 100 = \psi(L) = -\frac{I^2 L^2}{2\kappa A^2\sigma} + AL + B.$$

These imply that $A = \dfrac{I^2 L}{2\kappa A^2 \sigma}$, and $\psi(x) = 100 + \dfrac{I^2 x(L-x)}{2\kappa A^2\sigma}$. The function $V(x,t)$ will satisfy the PDE

$$\frac{\partial}{\partial t}(V + \psi) = k\frac{\partial^2}{\partial x^2}(V+\psi) + \frac{kI^2}{\kappa A^2\sigma} \quad \Longrightarrow \quad \frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

the boundary conditions

$$V(0,t) = U(0,t) - \psi(0) = 100 - 100 = 0, \quad t > 0,$$
$$V(L,t) = U(L,t) - \psi(L) = 100 - 100 = 0, \quad t > 0,$$

and the initial condition $\quad V(x,0) = U(x,0) - \psi(x) = 20 - \psi(x), \quad 0 < x < L.$

Separated functions $V(x,t) = X(x)T(t)$ satisfy the PDE and boundary conditions if $X(x)$ and $T(t)$ separately satisfy

$$X'' + \lambda X = 0, \quad 0 < x < L, \qquad T' + k\lambda T = 0, \quad t > 0.$$
$$X(0) = 0 = X(L);$$

The Sturm-Liouville system was discussed in Section 19.2. According to line 1 of Table 19.1, eigenvalues are $\lambda_n = n^2\pi^2/L^2$ and corresponding eigenfunctions are $X_n(x) = \sin(n\pi x/L)$. Since the auxiliary equation for the differential equation in $T(t)$ is $m + k\lambda_n = 0$, with solution $m = -k\lambda_n$, a general solution of the differential equation is $T(t) = be^{-k\lambda_n t} = be^{-n^2\pi^2 kt/L^2}$. Separated functions are $be^{-n^2\pi^2 kt/L^2} \sin \dfrac{n\pi x}{L}$. Because the PDE and boundary conditions are linear and homogeneous, we superpose separated functions in the form

$$V(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L}.$$

The initial condition requires

$$20-\psi(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad 0<x<L.$$

Since this is the eigenfunction expansion of $20-\psi(x)$ in terms of the $\sin(n\pi x/L)$,

$$b_n=\frac{2}{L}\int_0^L [20-\psi(x)]\sin\frac{n\pi x}{L}\,dx=-\left(\frac{160}{n\pi}+\frac{2I^2L^2}{\kappa A^2\sigma\pi^3n^3}\right)[1+(-1)^{n+1}].$$

The formal solution for temperature in the rod is therefore

$$\begin{aligned}U(x,t)&=\psi(x)+\sum_{n=1}^{\infty}-\left(\frac{160}{n\pi}+\frac{2I^2L^2}{\kappa A^2\sigma\pi^3n^3}\right)[1+(-1)^{n+1}]e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\\&=100+\frac{I^2x(L-x)}{2\kappa A^2\sigma}-\frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{80}{2n-1}+\frac{I^2L^2}{\kappa A^2\sigma\pi^2(2n-1)^3}\right]e^{-(2n-1)^2\pi^2kt/L^2}\sin\frac{(2n-1)\pi x}{L}.\end{aligned}$$

8. The initial boundary value problem for $y(x,t)$ is

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2}&=c^2\frac{\partial^2 y}{\partial x^2}-\frac{k}{\rho}, \quad 0<x<L, \quad t>0, \quad (k>0),\\y(0,t)&=0, \quad t>0,\\y(L,t)&=0, \quad t>0,\\y(x,0)&=f(x), \quad 0<x<L,\\y_t(x,0)&=g(x), \quad 0<x<L.\end{aligned}$$

Because the nonhomogeneity is time-independent, it may be removed by setting $y(x,t)=z(x,t)+\psi(x)$, where $\psi(x)$ is the static deflection defined by

$$\begin{aligned}c^2\frac{d^2\psi}{dx^2}-\frac{k}{\rho}&=0, \quad 0<x<L,\\\psi(0)=\psi(L)&=0.\end{aligned}$$

Integration of the differential equation gives $\psi(x)=\dfrac{kx^2}{2\rho c^2}+Ax+B$. The boundary conditions require

$$0=\psi(0)=B, \qquad 0=\psi(L)=\frac{kL^2}{2\rho c^2}+AL+B.$$

These imply that $A=-\dfrac{kL}{2\rho c^2}$, and $\psi(x)=-\dfrac{kx(L-x)}{2\rho c^2}$. The function $z(x,t)$ will satisfy the PDE

$$\frac{\partial^2}{\partial t^2}(z+\psi)=c^2\frac{\partial^2}{\partial x^2}(z+\psi)-\frac{k}{\rho} \quad\Longrightarrow\quad \frac{\partial^2 z}{\partial t^2}=c^2\frac{\partial^2 z}{\partial x^2}, \quad 0<x<L, \quad t>0,$$

the boundary conditions

$$\begin{aligned}z(0,t)&=y(0,t)-\psi(0)=0, \quad t>0,\\z(L,t)&=y(L,t)-\psi(L)=0, \quad t>0,\end{aligned}$$

and the initial conditions

$$\begin{aligned}z(x,0)&=y(x,0)-\psi(x)=f(x)+\frac{kx(L-x)}{2\rho c^2}, \quad 0<x<L,\\z_t(x,0)&=y_t(x,0)-d\psi/dt=g(x), \quad 0<x<L.\end{aligned}$$

Separated functions $z(x,t) = X(x)T(t)$ satisfy the PDE and boundary conditions if $X(x)$ and $T(t)$ separately satisfy

$$X'' + \lambda X = 0, \quad 0 < x < L, \qquad X(0) = 0 = X(L); \qquad\qquad T'' + \lambda c^2 T = 0, \quad t > 0.$$

The Sturm-Liouville system was discussed in Section 19.2. According to line 1 of Table 19.1, eigenvalues are $\lambda_n = n^2\pi^2/L^2$ and corresponding eigenfunctions are $X_n(x) = \sin(n\pi x/L)$. Since the auxiliary equation for the differential equation in $T(t)$ is $m^2 + c^2\lambda_n = 0$, with solution $m = \pm c\sqrt{\lambda_n}i = \pm n\pi ci/L$, a general solution of the differential equation is $T(t) = A\cos\frac{n\pi ct}{L} + B\sin\frac{n\pi ct}{L}$. Separated functions are $\left(A\cos\frac{n\pi ct}{L} + B\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$. Because the PDE and boundary conditions are linear and homogeneous, we superpose separated functions in the form

$$z(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}.$$

The first initial condition requires

$$f(x) + \frac{kx(L-x)}{2\rho c^2} = \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{L}, \quad 0 < x < L,$$

and therefore the A_n are coefficients in the Fourier sine series of the odd, $2L$-periodic extension of the function on the left,

$$A_n = \frac{2}{L}\int_0^L \left[f(x) + \frac{kx(L-x)}{2\rho c^2}\right]\sin\frac{n\pi x}{L}dx.$$

The second condition gives

$$g(x) = \sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\frac{n\pi x}{L}, \quad 0 < x < L,$$

and hence the $(n\pi c/L)B_n$ are coefficients in the Fourier sine series of the odd, $2L$-periodic extension of the function $g(x)$,

$$\frac{n\pi c}{L}B_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx \qquad \Longrightarrow \qquad B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}dx.$$

The formal solution is therefore

$$y(x,t) = -\frac{kx(L-x)}{2\rho c^2} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L},$$

where A_n and B_n are defined above.

10. The initial boundary value problem for $y(x,t)$ is

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2} - g, \quad 0 < x < L, \quad t > 0,\\ y(0,t) &= 0, \quad t > 0,\\ y_x(L,t) &= F_0/\tau, \quad t > 0,\\ y(x,0) &= 0, \quad 0 < x < L,\\ y_t(x,0) &= 0, \quad 0 < x < L.\end{aligned}$$

Because the nonhomogeneities are time-independent, they may be removed by setting $y(x,t) = z(x,t) + \psi(x)$, where $\psi(x)$ is the static deflection defined by

$$c^2\frac{d^2\psi}{dx^2} - g = 0, \quad 0 < x < L,$$
$$\psi(0) = 0,$$
$$\psi'(L) = F_0/\tau.$$

Integration of the differential equation gives $\psi(x) = \dfrac{gx^2}{2c^2} + Ax + B$. The boundary conditions require

$$0 = \psi(0) = B, \qquad \frac{F_0}{\tau} = \psi'(L) = \frac{gL}{c^2} + A.$$

These imply that $A = \dfrac{F_0}{\tau} - \dfrac{gL}{c^2}$, and $\psi(x) = -\dfrac{gx(x-2L)}{2c^2} + \dfrac{F_0 x}{\tau}$. The function $z(x,t)$ will satisfy the PDE

$$\frac{\partial^2}{\partial t^2}(z+\psi) = c^2\frac{\partial^2}{\partial x^2}(z+\psi) - g \quad \Longrightarrow \quad \frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

the boundary conditions

$$z(0,t) = y(0,t) - \psi(0) = 0, \quad t > 0,$$
$$z_x(L,t) = y_x(L,t) - \psi'(L) = F_0/\tau - F_0/\tau = 0, \quad t > 0,$$

and the initial conditions

$$z(x,0) = y(x,0) - \psi(x) = \frac{gx(x-2L)}{2c^2} - \frac{F_0 x}{\tau}, \quad 0 < x < L,$$
$$z_t(x,0) = y_t(x,0) = 0, \quad 0 < x < L.$$

Separated functions $z(x,t) = X(x)T(t)$ satisfy the PDE, the boundary conditions, and the second initial condition if $X(x)$ and $T(t)$ separately satisfy

$$X'' + \lambda X = 0, \quad 0 < x < L, \qquad\qquad T'' + \lambda c^2 T = 0, \quad t > 0,$$
$$X(0) = 0 = X'(L); \qquad\qquad T'(0) = 0.$$

The Sturm-Liouville system was discussed in Section 19.2. According to line 3 of Table 19.1, eigenvalues are $\lambda_n = \dfrac{(2n-1)^2\pi^2}{4L^2}$ and corresponding eigenfunctions are $X_n(x) = \sin\dfrac{(2n-1)\pi x}{2L}$. Since the auxiliary equation $m^2 + c^2\lambda_n = 0$ for the differential equation in $T(t)$ has solution $m = \pm c\sqrt{\lambda_n}i = \dfrac{\pm(2n-1)\pi ci}{2L}$, a general solution of the differential equation is $T(t) = A\cos\dfrac{(2n-1)\pi ct}{2L} + B\sin\dfrac{(2n-1)\pi ct}{2L}$. The condition $T'(0) = 0$ requires $B = 0$. Separated functions are $A\cos\dfrac{(2n-1)\pi ct}{2L}\sin\dfrac{(2n-1)\pi x}{2L}$. Because the PDE, boundary conditions, and second initial condition are linear and homogeneous, we superpose separated functions in the form

$$z(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{(2n-1)\pi ct}{2L}\sin\frac{(2n-1)\pi x}{2L}.$$

The first initial condition requires

$$\frac{gx(x-2L)}{2c^2} - \frac{F_0 x}{\tau} = \sum_{n=1}^{\infty} A_n \sin\frac{(2n-1)\pi x}{2L}, \quad 0 < x < L,$$

and therefore the A_n are coefficients in the eigenfunction expansion of the function on the left,

$$A_n = \frac{2}{L}\int_0^L \left[\frac{gx(x-2L)}{2c^2} - \frac{F_0 x}{\tau}\right] \sin\frac{(2n-1)\pi x}{2L} dx.$$

Integration by parts leads to

$$A_n = \frac{-16L^2 g}{(2n-1)^3\pi^2 c^2} + \frac{8LF_0(-1)^n}{(2n-1)^2\pi^2\tau}.$$

The formal solution is therefore

$$\begin{aligned} y(x,t) &= -\frac{gx(x-2L)}{2c^2} + \frac{F_0 x}{\tau} + \sum_{n=1}^{\infty}\left[\frac{-16L^2 g}{(2n-1)^3\pi^3 c^2} + \frac{8LF_0(-1)^n}{(2n-1)^2\pi^2\tau}\right] \cos\frac{(2n-1)\pi ct}{2L} \sin\frac{(2n-1)\pi x}{2L} \\ &= -\frac{gx(x-2L)}{2c^2} + \frac{F_0 x}{\tau} + \frac{8L}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{-2Lg}{(2n-1)^3\pi c^2} + \frac{F_0(-1)^n}{(2n-1)^2\tau}\right] \cos\frac{(2n-1)\pi ct}{2L} \sin\frac{(2n-1)\pi x}{2L}. \end{aligned}$$