10 **1.** Evaluate the line integral

$$\int_C 32y\,ds$$

where C is the curve $x = y^2$, $z = 4 + x$ from the point $(4, 2, 8)$ to the point $(1, 1, 5)$.

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1. A use $(-C)$:
Let $y = t$, $x = t^2$, $z = 4 + t^2$; $1 \le t \le 2$.

$$\int_C 32y\,ds = \int_{-C} 32y\,ds = \int_1^2 32t\sqrt{x'^2 + y'^2 + z'^2}\,dt$$

$$= \int_1^2 32t\sqrt{4t^2 + 1 + 4t^2}\,dt = \int_1^2 32t\sqrt{1 + 8t^2}\,dt$$

$$= \frac{4}{3}(1 + 8t^2)^{3/2}\Big|_{t=1}^{2} = \frac{4}{3}\left(33^{3/2} - 9^{3/2}\right)$$

$$= \frac{4}{3}(33\sqrt{33} - 27) = 4(11\sqrt{33} - 9).$$

B. Use C but use $y = -t$:
Let $y = -t$, $x = t^2$, $z = 4 + t^2$; $-2 \le t \le -1$.

$$I = \int_C 32y\,ds = \int_{-2}^{-1} -32t\sqrt{x'^2 + y'^2 + z'^2}\,dt$$

$$= \int_{-1}^{-2} 32t\sqrt{4t^2 + 1 + 4t^2}\,dt = \int_{-1}^{-2} 32t\sqrt{1 + 8t^2}\,dt$$

$$= \frac{4}{3}(1 + 8t^2)^{3/2}\Big|_{-1}^{-2} = \frac{4}{3}\left(33^{3/2} - 9^{3/2}\right)$$

$$= 44\sqrt{33} - 36 \quad \text{(as above)}.$$

Note: For $\int_{\pm 1}^{\pm 2} 32t\sqrt{1 + 8t^2}\,dt$, let $u = 1 + 8t^2$.
Then, $du = 16t\,dt$, $I = \int_{u=9}^{33} 2\sqrt{u}\,du = \frac{4}{3}u^{3/2}\Big|_{u=9}^{33}$

$$= \frac{4}{3}\left(33^{3/2} - 9^{3/2}\right) = 44\sqrt{33} - 36.$$

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7 **2.** Evaluate the line integral

$$\int_C \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy,$$

where C consists of the three straight line segments joining successively the points $(2,0)$, $(2,1)$, $(-2,1)$, and $(-2,0)$.

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2. Let $\vec{F} = \left(\frac{x}{x^2+y^2}\right)\vec{i} + \left(\frac{y}{x^2+y^2}\right)\vec{j}$.

Then $\vec{F} = \nabla\phi$ where $\phi = \frac{1}{2}\ln(x^2+y^2)$, $(x,y) \neq (0,0)$.

$I = \int_C \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy = \int_C \vec{F}\cdot d\vec{r} = \phi\big|_C$

The integral is path-independent on $\mathbb{R}^2 \setminus \{(0,0)\}$

$I = \phi(-2,0) - \phi(2,0) = \frac{1}{2}\ln 4 - \frac{1}{2}\ln 4 = 0$.

Alternatively, since I is path-independent on $\mathbb{R}^2 \setminus \{(0,0)\}$ replace C with a semi-circle : $x^2+y^2=4$, $y \geq 0$.

$x = 2\cos t$, $y = 2\sin t$, $0 \leq t \leq \pi$

$I = \int_C \frac{1}{4}x\,dx + \frac{1}{4}y\,dy = \int_C \frac{1}{8}\nabla(x^2+y^2)\cdot d\vec{r}$

$= \frac{1}{8}(x^2+y^2)\big|_C = \frac{4}{8} - \frac{4}{8} = 0$.

OR

$I = \int_0^{2\pi}\left[\frac{2}{4}\cos t\,(-\sin t) + \frac{2}{4}\sin t\cos t\right]dt = 0$.

9 **3.** Evaluate the surface integral

$$\oiint_S [(x^2+y^2)\hat{\mathbf{i}} + x^3\hat{\mathbf{j}} + (3z+\sin x)\hat{\mathbf{k}}] \cdot \hat{\mathbf{n}}\, dS,$$

where S is the closed surface that bounds the volume enclosed by the surfaces

$$x^2+y^2=4, \quad z=1, \quad z=-1,$$

and $\hat{\mathbf{n}}$ is the unit inward pointing normal to S.

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3. Using the Divergence Theorem, since $\vec{n}$ points inward,

The integral, $I = -\iiint_V \operatorname{div} \vec{F}\, dV$

$= -\iiint_V (2x+3)\, dV = -\iiint_V 3\, dV$, (since x is odd on V and V is symmetric in x).

$= -3\,\text{Volume}(V) = -3\pi R^2 H$, V is a cylinder

$= -3\pi(4)(2) = -24\pi$.

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or $I = -\int_0^{2\pi}\int_0^2\int_{-1}^1 (2r\cos\theta + 3)\, dz\, r\, dr\, d\theta$

$= -\int_0^{2\pi}\int_0^2 2(2r^2\cos\theta + 3r)\, dr\, d\theta$

$= -\int_0^{2\pi} \left(\frac{4r^3}{3}\cos\theta + 3r^2\right)\Big|_0^2 d\theta$

$= -\int_0^{2\pi} \left(\frac{32}{3}\cos\theta + 12\right) d\theta$

$= -\left(\frac{32}{3}\sin\theta + 12\theta\right)\Big|_0^{2\pi} = -24\pi$

Alternate Solution

3\. Note: $S = S_1$ (curved sides) $\cup$ S_2 (top) $\cup$ S_3 (bottom).

Without Divergence Theorem (Note: This method is *much* harder).

$S_1: x^2+y^2=4$, grad $(x^2+y^2) = (2x, 2y, 0)$

$$\hat{n}_1 = \frac{-(x,y,0)}{\sqrt{x^2+y^2}} = \frac{-(x,y,0)}{2}$$

$y^2 = 4-x^2$: $\frac{\partial}{\partial x}$: $2y\,y_x = -2x$, $y_x = \frac{-x}{y}$; $y_z = 0$

$$dS = \sqrt{1+y_x^2+y_z^2}\,dA = \sqrt{1+\frac{x^2}{y^2}}\,dA = \sqrt{\frac{y^2+x^2}{y^2}}\,dA = \frac{2}{|y|}\,dA$$

Project S_1^+ $(y \geq 0)$ and S_1^- $(y \leq 0)$ onto the xz plane.

$$I_1 = \iint_{S_1} \vec{F}\cdot\hat{n}_1\,dS = \iint_{S_1^+} \vec{F}\cdot\hat{n}_1\,dS + \iint_{S_1^-} \vec{F}\cdot\hat{n}_1\,dS$$

$$\vec{F} = (x^2+y^2)\hat{i} + x^3\hat{j} + (3z+\sin x)\hat{k}$$

$$\vec{F}\cdot\hat{n}_1 = -\frac{1}{2}\left[x^3 + xy^2 + x^3 y\right]$$

Since this is an odd function of x and S_1 is symmetric with respect to x, $I_1 = 0$.

Let S_2 be the top $(z=1)$ and S_3 the bottom $(z=-1)$.

$$I = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} \qquad \hat{n}_2 = -\hat{k},\ \hat{n}_3 = \hat{k}$$

$$(S_2)_{xy} = (S_3)_{xy}$$

$$= 0 + \iint_{(S_2)_{xy}} -(3z+\sin x)\Big|_{z=1} + (3z+\sin x)\Big|_{z=-1}\,dA_{xy}$$

$$= \iint_{x^2+y^2\leq 4} -6\,dA = -6 \text{ Area of disk}$$

$$= -6\,\pi(4) = -24\,\pi.$$

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 4. Evaluate the surface integral

$$\iint_S (y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot \hat{\mathbf{n}}\, dS$$

where S is that part of the surface $z = 9 - x^2 - y^2$ above the xy-plane, and $\hat{\mathbf{n}}$ is the unit normal to S with positive z-component.

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4. $S:\ G = z + x^2 + y^2 = 9$, $\nabla G = (2x, 2y, 1)$

$\hat{n} = \dfrac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}}$. $\underline{S \text{ intersects the } xy \text{ plane on } x^2 + y^2 = 9}$

From $z = 9 - x^2 - y^2$, $z_x = -2x$, $z_y = -2y$

$dS = \sqrt{1 + z_x^2 + z_y^2}\, dA = \sqrt{1 + 4x^2 + 4y^2}\, dA$

$$I = \iint_{S_{xy}} (y, -x, z) \cdot \frac{(2x, 2y, 1)}{\sqrt{4x^2 + 4y^2 + 1}} \Bigg|_{z = 9 - x^2 - y^2} \sqrt{1 + 4x^2 + 4y^2}\, dA$$

$$= \iint_{x^2 + y^2 \le 9} (2xy - 2xy + z)\Big|_{z = 9 - x^2 - y^2} dA$$

$$= \iint_{x^2 + y^2 \le 9} 9 - x^2 - y^2 \ dx\, dy$$

Using polar coordinates,

$$I = \int_0^{2\pi} \int_0^3 (9 - r^2)\, r\, dr\, d\theta$$

$$= 2\pi \left. \frac{(9 - r^2)^2}{2(-2)} \right|_0^3 \quad \underline{\underline{\text{OR}}} \quad 2\pi \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_0^3$$

$$= \frac{\pi}{-2}(0 - 81) \qquad\qquad = 2\pi \left[\frac{81}{2} - \frac{81}{4} \right]$$

$$= \frac{81\pi}{2}.$$

Alternate Solution

4. Using the Divergence Theorem, we first "add" and "subtract" the base, S_1 : $x^2+y^2 \le 9$, $z=0$.

$$I = \iint_{S \cup S_1} - \iint_{S_1}$$

$\hat{n}$ = unit outward normal to $S \cup S_1$.

Div. Thm

on S_1, $\hat{n}_1 = -\hat{k}$

$$I = \iiint_V \operatorname{div} \vec{F} \, dV - \iint_{S_1} \vec{F} \cdot (-\hat{k}) \, dA$$

$$= \iiint_V 1 \, dV + \iint_{S_1} z \Big|_{z=0} dA$$

$$= \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} dz \, r \, dr \, d\theta$$

$$= 2\pi \int_0^3 (9-r^2) \, r \, dr$$

$$= 2\pi \frac{(9-r^2)^2}{2(-2)} \Big|_0^3 \quad \underline{\underline{OR}} \quad 2\pi \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_0^3$$

$$= \frac{81\pi}{2}.$$