Math 3132 **Test # 1 Solutions** **October 16, 2008**

Note: There are 4 questions for a total of 40 marks.

Values

(9) 1. (a) Let $\mathbf{F}(x,y,z) = (y e^{z})\hat{\mathbf{i}} + (x e^{z} - 1)\hat{\mathbf{j}} + (x y e^{z} + z)\hat{\mathbf{k}}$. Calculate $\nabla \times \mathbf{F}$, showing details.

(b) Find a function φ such that $\nabla\varphi = \mathbf{F}$.

(c) Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the straight line from (−1, 1, 0) to (1, 2, 1).

Solution: (a) $$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (y e^{z}) & (x e^{z} - 1) & (x y e^{z} + z) \end{vmatrix} = (x e^{z} - x e^{z})\hat{\mathbf{i}} - (y e^{z} - y e^{z})\hat{\mathbf{j}} + (e^{z} - e^{z})\hat{\mathbf{k}} = \mathbf{0}.$$

(b) By inspection, $\varphi = x y e^{z} - y + \dfrac{z^2}{2} + (K)$. The constant K is optional. Then, $\nabla\varphi = \mathbf{F}$.

(c) $$\int_C \mathbf{F}\cdot d\mathbf{r} = \varphi\Big|_C = \varphi(1,2,1) - \varphi(-1,1,0) = (2e - 2 + \frac{1}{2}) - (-1-1)$$
$$= 2e + \frac{1}{2} = \frac{4e+1}{2}.$$

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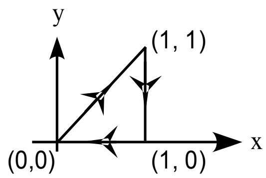
(11) 2. Consider the closed line integral $I = \oint_C x^2 y\,dx + y^2 x\,dy$ where C consists of the three straight line segments, first from (0, 0) to (1, 1), then from (1, 1) to (1, 0), and finally from (1, 0) to (0, 0).

(a) Evaluate the above integral I by ***parametrizing*** each part of C in terms of a parameter, t.

(b) Evaluate the above integral I by ***using Green's Theorem***.

Solution:



2. (a) (0, 0) to (1, 1): $x = y = t;\ 0 \le t \le 1;$

(1, 1) to (1, 0): $x = 1,\ y = t;\ 0 \le t \le 1$ (*reversed*);

(1, 0) to (0, 0): $x = t,\ y = 0;\ 0 \le t \le 1$ (*reversed*).

$$I = \int_0^1 t^3 + t^3\,dt - \int_0^1 t^2\,dt = \left(\frac{t^4}{2} - \frac{t^3}{3}\right)\Bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

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2. (b) $P = x^2 y,\ Q = y^2 x$. Since the curve is traversed *clockwise*, we multiply the integral on the right-hand side by -1 when using Green's Theorem. Let R be the interior of the given triangle.

$$I = -\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = -\iint_R \left(y^2 - x^2\right) dA = \int_0^1\int_0^x x^2 - y^2\,dy\,dx$$

$$= \int_0^1 \left(x^2 y - \frac{y^3}{3}\right)\Bigg|_{y=0}^{x} dx = \int_0^1 x^3 - \frac{x^3}{3}\,dx = \left(\frac{2}{3}\frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{6}.$$

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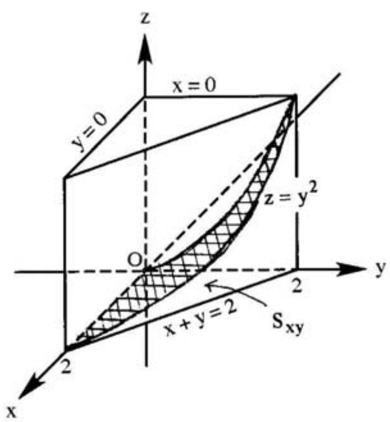
(11) 3. Calculate the integral: $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$, where $\mathbf{F} = (x^2)\hat{\mathbf{i}} + (2y)\hat{\mathbf{j}} + (z)\hat{\mathbf{k}}$, S is that part of the surface $z = y^2$ that lies inside the first octant and is bounded by the vertical surfaces $x = 0$, $y = 0$ and $x + y = 2$, and $\hat{\mathbf{n}}$ is the *upward* pointing unit normal to the surface S.
Remark: S consists of only one piece; it is part of $z = y^2$ as described above.

Solution:



$G = z - y^2 = 0$. $\nabla G = (0,-2y,1)$. $\hat{\mathbf{n}} = \dfrac{(0,-2y,1)}{\sqrt{4y^2+1}}$.

Project onto the x-y-plane. Then, S_{xy} is the triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 2$.

$$dS = \sqrt{1+(z_x)^2+(z_y)^2}\;dA = \sqrt{1+4y^2}\;dA$$

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_{xy}} \left.\frac{-4y^2+z}{\sqrt{4y^2+1}}\right|_{z=y^2} \sqrt{1+4y^2}\;dydx = \int_0^2\int_0^{2-x} -3y^2\,dydx$$

$$= \int_0^2 \left(-y^3\Big|_{y=0}^{2-x} \right) dx = \int_0^2 -(2-x)^3\,dx = \left.\frac{(2-x)^4}{4}\right|_0^2 = -4.$$

Alternatively, with the order of integration interchanged:

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_0^2\int_0^{2-y} -3y^2\,dx\,dy = \int_0^2 \left(-3xy^2\Big|_{x=0}^{2-y} \right) dy = \int_0^2 -3(2-y)y^2\,dy = \int_0^2 -6y^2+3y^3\,dy$$

$$= \left.\left(-2y^3 + \frac{3y^4}{4} \right)\right|_0^2 = (-16+12) = -4.$$

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(9) 4. Let S be the *closed* surface enclosing the volume bounded by the upper hemisphere $x^2 + y^2 + z^2 = 4$ with $z \geq 0$ and the x-y plane.

Calculate the closed surface integral: $\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$, where $\mathbf{F} = (z^3)\hat{\mathbf{i}} + (x^2)\hat{\mathbf{j}} + (z^2+4)\hat{\mathbf{k}}$, and $\hat{\mathbf{n}}$ is the unit *inner* normal to the surface S.

Solution: Since $\hat{\mathbf{n}}$ is the unit *inner* normal, we multiply the integral on the right-hand side by -1 when using the Divergence Theorem. Then we use cylindrical coordinates.

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = -\iiint_V \nabla \cdot \mathbf{F}\, dV = -\int_0^{2\pi}\int_0^2\int_0^{\sqrt{4-r^2}} 2z\,dz\, r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_0^2 \left[z^2 \Big|_0^{\sqrt{4-r^2}} \right] r\,dr\,d\theta = -\int_0^{2\pi}\int_0^2 (4-r^2)\,r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_0^2 (4r - r^3)\,dr\,d\theta = -\int_0^{2\pi} \left(2r^2 - \frac{r^4}{4} \right)\Bigg|_0^2 d\theta$$

$$= -2\pi(8-4) = -8\pi.$$

Alternative Solutions: **These solutions are generally much more complicated and are therefore NOT RECOMMENDED! They are shown here only for comparisons.**

1 (c) Using a parametrization: $x = -1 + 2t,\ y = 1 + t,\ z = t;\ 0 \le t \le 1;$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (ye^z)\,dx + (xe^z - 1)\,dy + (xye^z + z)\,dz$$

$$= \int_0^1 (1+t)e^t 2 + \left[(-1+2t)e^t - 1\right] + \left[(-1+2t)(1+t)e^t + t\right] dt$$

$$= \int_0^1 \left[(5t + 2t^2)e^t - 1 + t\right] dt.$$

Integrate by parts twice:

$$\int \left[(5t+2t^2)e^t\right] dt = \int (5t+2t^2)\,d(e^t) = (5t+2t^2)e^t - \int e^t(5+4t)\,dt$$

$$= (5t+2t^2)e^t - \int (5+4t)\,d(e^t)$$

$$= (5t+2t^2)e^t - \left[(5+4t)e^t - \int e^t(4)\,dt\right]$$

$$= (5t+2t^2)e^t - \left[(5+4t)e^t - 4e^t\right]$$

$$= (2t^2 + t - 1)e^t.$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \left[(2t^2 + t - 1)e^t - t + \frac{t^2}{2}\right]_0^1 = \left[2e - 1 + \frac{1}{2}\right] - [-1]$$

$$= 2e + \frac{1}{2} = \frac{4e+1}{2}.$$

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3. We might be tempted to try to use the Divergence Theorem, but **DON'T**! We would have to "add" and "subtract" faces (surfaces) on $x = 0$, $x + y = 2$, and $z = 0$.
   This is a lot of work and would take MUCH MORE work than the original solution!

Alternative Solution:

4. Using the Divergence Theorem and spherical coordinates,

(see D. Trim, *Calculus for Engineers,* 4th Edition, section 14.12),

$$\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\iiint_V \nabla\cdot\mathbf{F}\,dV = -\iiint_V 2z\,dV = -\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 (2r\cos\phi)\,r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$= -\int_0^{2\pi}\int_0^{\pi/2}\int_0^2 r^3\sin(2\phi)\,dr\,d\phi\,d\theta = -\int_0^{2\pi}\int_0^{\pi/2}\left(\frac{r^4}{4}\right)\Bigg|_0^2 \sin(2\phi)\,d\phi\,d\theta$$

$$= -4\int_0^{2\pi}\left(\frac{-\cos(2\phi)}{2}\right)\Bigg|_0^{\pi/2} d\theta = -4\int_0^{2\pi}\left(\frac{-\cos(\pi)+\cos(0)}{2}\right)d\theta = -4\int_0^{2\pi}(1)\,d\theta$$

$$= -8\pi.$$

Alternative Solution:

4. Without using the Divergence Theorem. Let S_1 be the upper hemisphere and S_2 its base in the x-y plane.

$$\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS + \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS.$$

For S_1, $G = x^2+y^2+z^2 = 4$, $\frac{1}{2}\nabla G = (x,y,z)$, and $\hat{\mathbf{n}} = -\dfrac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = -\dfrac{(x,y,z)}{\sqrt{4}} = -\dfrac{(x,y,z)}{2}$.

Differentiate $x^2+y^2+z^2=4$ with respect to x: $2x + 2zz_x = 0$, $z_x = -\dfrac{x}{z}$; $z_y = -\dfrac{y}{z}$.

$$\sqrt{1+(z_x)^2+(z_y)^2} = \sqrt{1+(-\frac{x}{z})^2+(-\frac{y}{z})^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \sqrt{\frac{z^2+x^2+y^2}{z^2}} = \sqrt{\frac{4}{z^2}} = \frac{2}{z}.$$

For S_2, $\hat{\mathbf{n}} = \hat{\mathbf{k}} = (0,0,1)$ and $dS = dA$.

$$\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS + \iint_{S_2} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

$$= -\iint_{(S_1)_{xy}} \left.\frac{\left[xz^3 + x^2y + z(z^2+4)\right]}{2}\,\frac{2}{z}\right|_{z=\sqrt{4-(x^2+y^2)}} dA + \iint_{(S_2)_{xy}} (z^2+4)\Big|_{z=0}\,dA.$$

$$(S_1)_{xy} = (S_2)_{xy} = \left\{ (x,y):\ x^2+y^2 \le 4 \right\}.$$

In the first integral, $(S_1)_{xy}$ is symmetric in x and y and x and y are odd functions. Thus, the first two terms in that integral are zero. Using polar coordinates,

$$\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\iint_{(S_1)_{xy}} 8-(x^2+y^2)\,dA + \iint_{(S_2)_{xy}} 4\,dA \quad (*)$$

$$= -\int_0^{2\pi}\int_0^2 (8-r^2)\,r\,dr\,d\theta + 4\,\text{Area}((S_2)_{xy})$$

$$= -2\pi\left(4r^2 - \frac{r^4}{4}\right)\Bigg|_0^2 + 4(\pi 4)$$

$$= -2\pi(16-4) + 4(\pi 4)$$

$$= -8\pi.$$

Alternatively, from (*) above, $\displaystyle\oiint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int_0^{2\pi}\int_0^2 \left[-(8-r^2)+4\right] r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[r^3 - 4r\right] dr\,d\theta$

$$= 2\pi\left(\frac{r^4}{4} - 2r^2\right)\Bigg|_0^2 = 2\pi(4-8) = -8\pi.$$
