Hyperbolic Geometry

Recall Euclid's Fifth Postulate:
For every line ℓ and point P that does not lie on ℓ, there exists a unique line m through P and parallel to ℓ.

There are two ways that can be different, we could replace 'a unique' with 'many' or 'none'.

Hyperbolic Geometry

Hyperbolic Fifth Postulate:
Given a line ℓ and a point P not on ℓ, there are many lines on P parallel to ℓ.

★ - In this context, parallel MEANS the lines do not meet.

Poincaré Model of a Hyperbolic Geometry

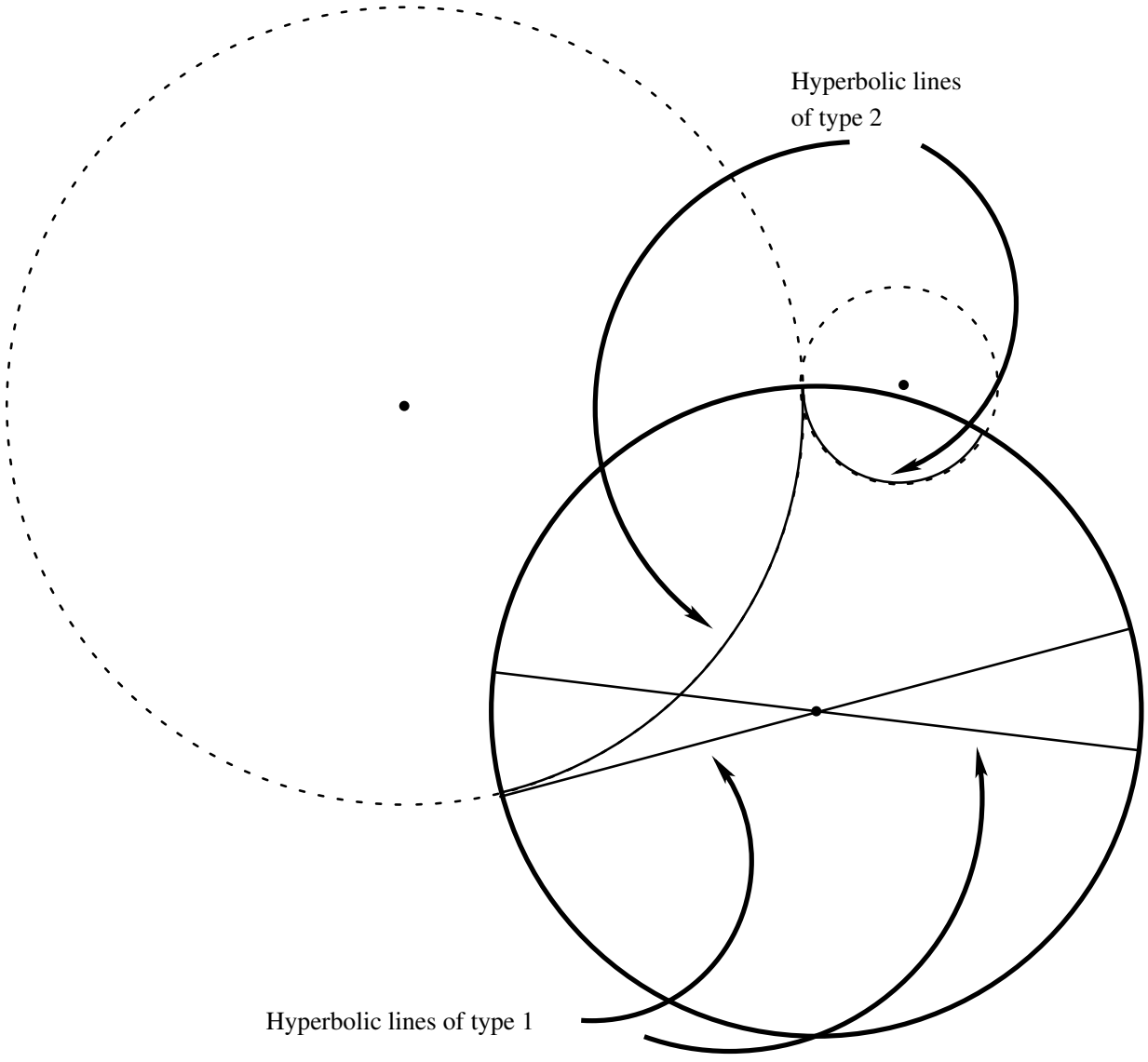
Given H, a circle with center O:

Points of the geometry are all the points that are STRICTLY inside of H. This does not include the points on the circle itself.

Lines of the geometry are of two types:

- diameters
- Parts of circles that are perpendicular to H

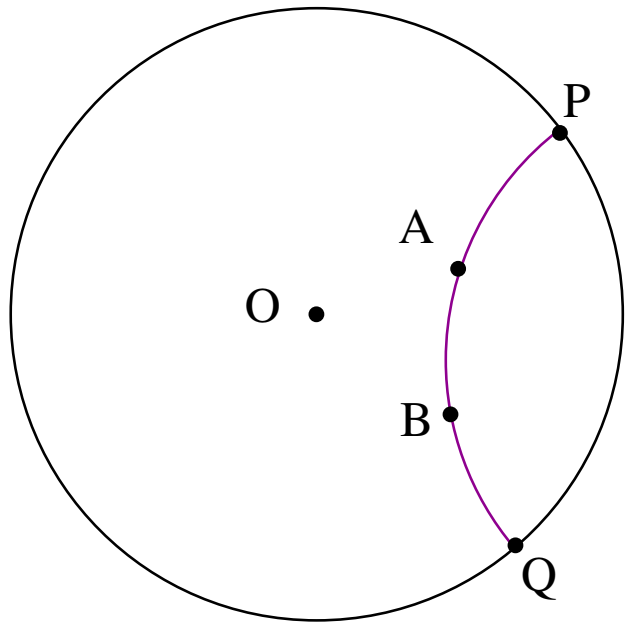
Poincaré Model



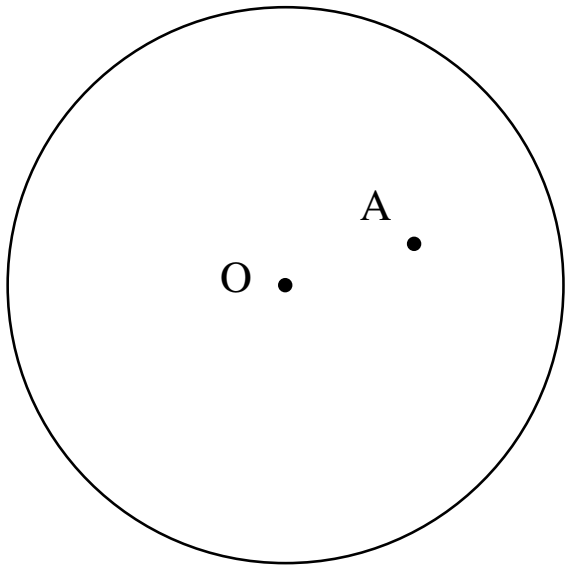
Hyperbolic distance

The formula for the distance between two hyperbolic points is

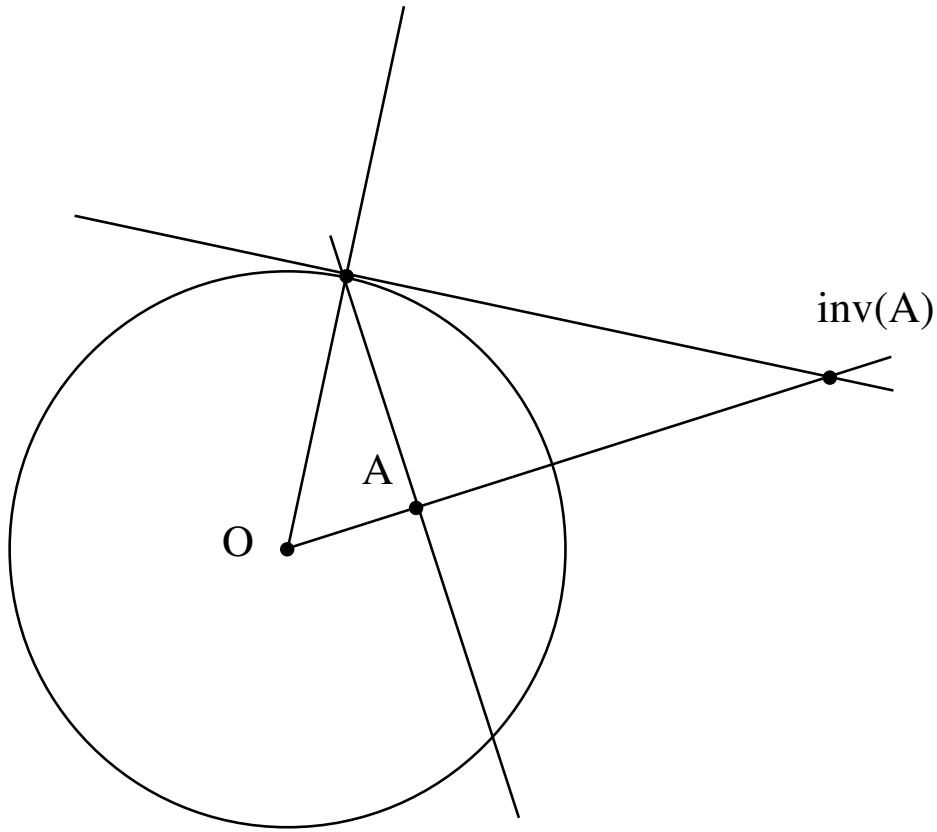
$$d_H(A,B) = \left| \ln \frac{d(A,P)\cdot d(B,Q)}{d(A,Q)\cdot d(B,P)} \right| .$$



Recall Circle Inversion

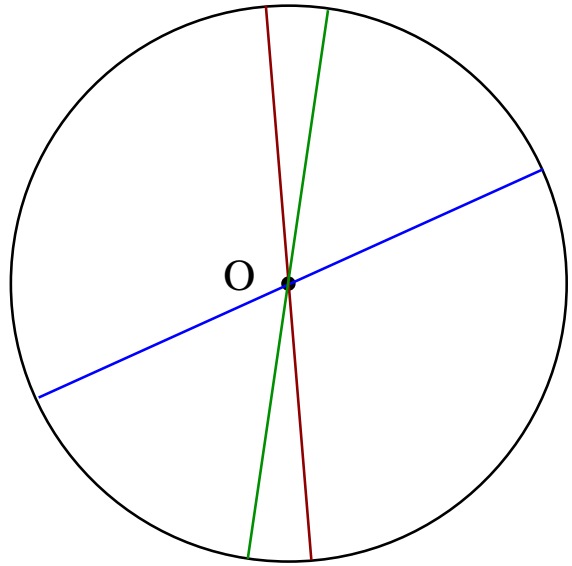


Recall Circle Inversion



Hyperbolic Lines through O, the center of H

All hyperbolic lines that pass thought O are diameters.



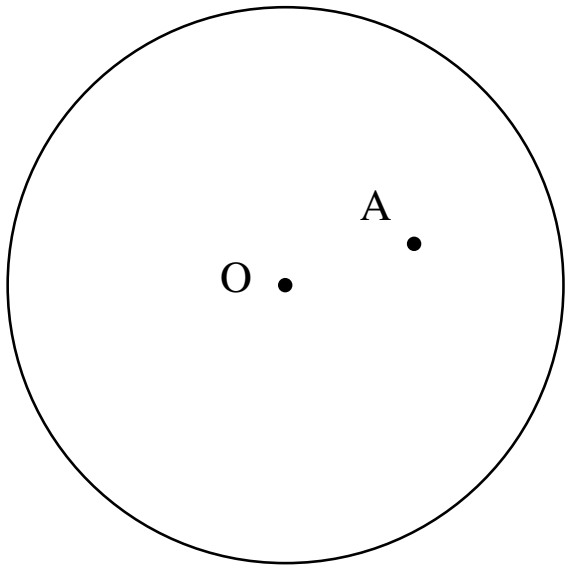
Hyperbolic Lines through A, a point other than the center

The perpendicular bisector of the line $A\,\mathrm{inv}(A)$ is called the center line for A. We sometimes denote this ℓ_A.

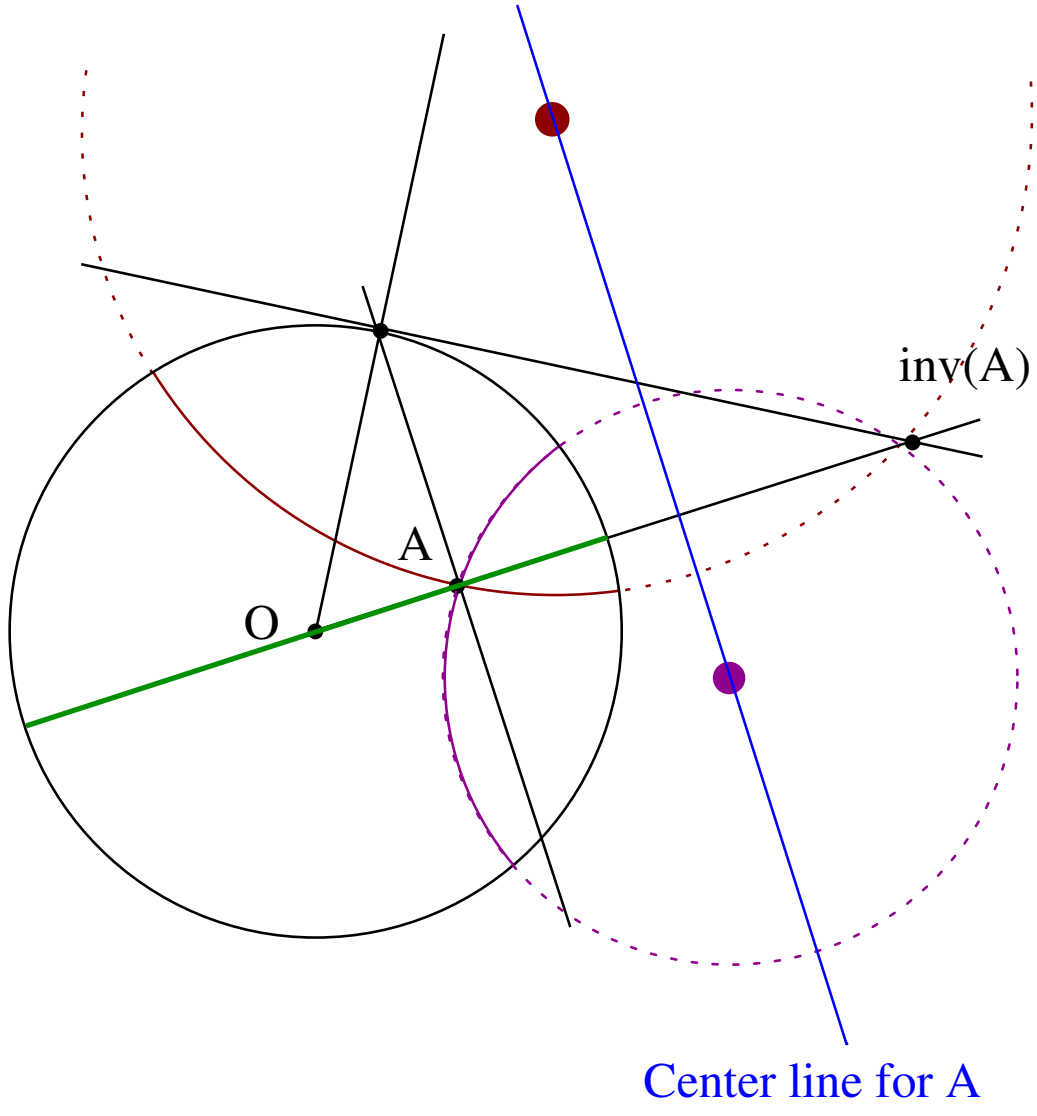
All of the lines through A are:

- The diameter that passes through A.
- The part interior to H of a circle that has center on ℓ_A and passes through the point A.

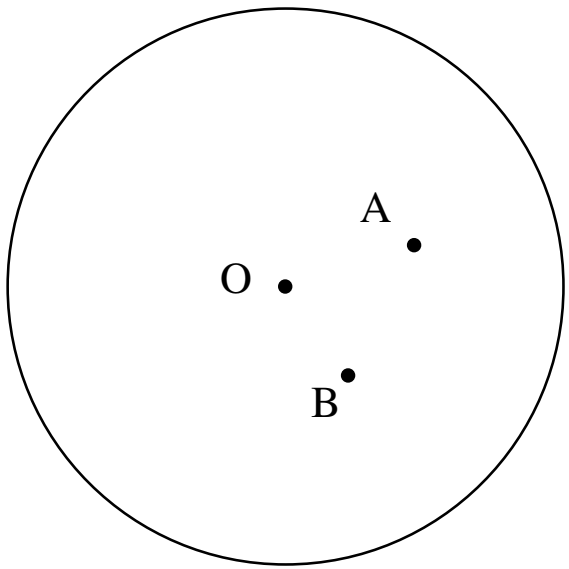
Hyperbolic Lines through A



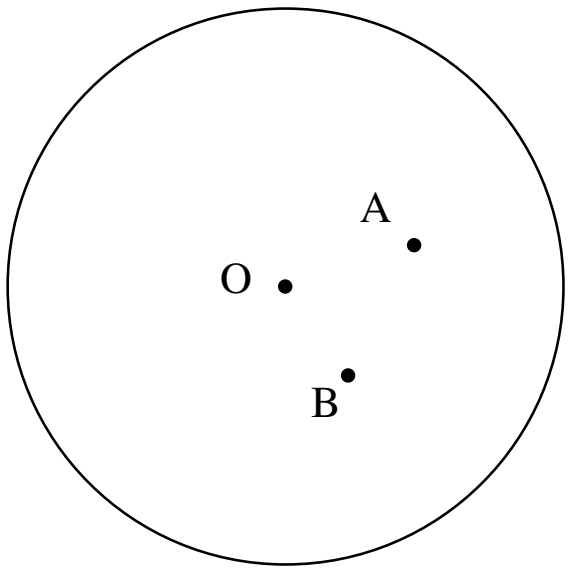
Hyperbolic Lines through A



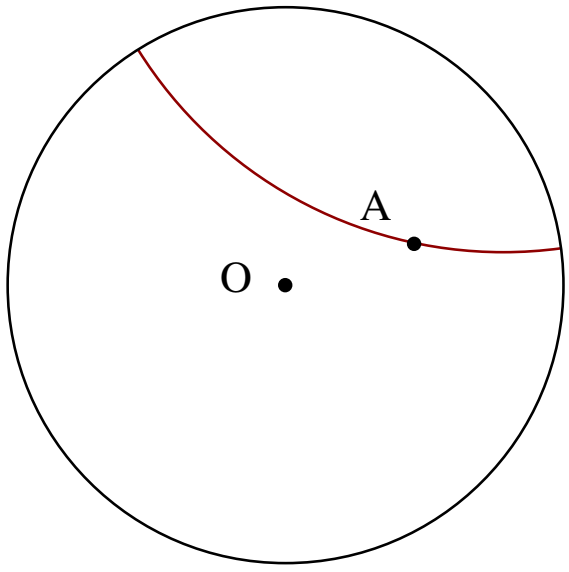
The Hyperbolic line passing through A and B, (Construction 2)



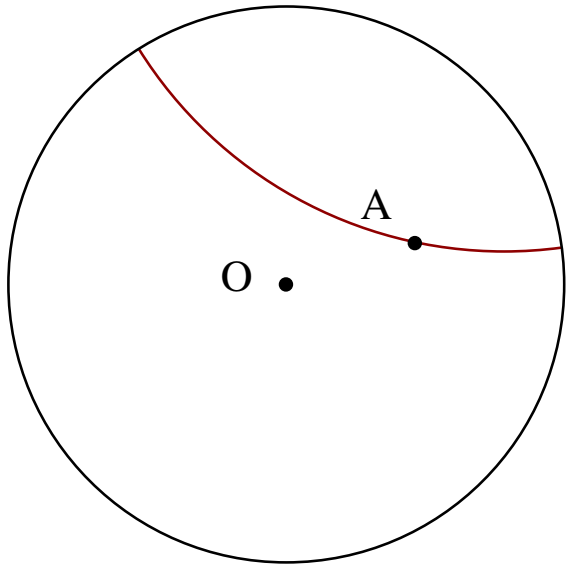
The Hyperbolic line passing through A and B, (Alternate Method)

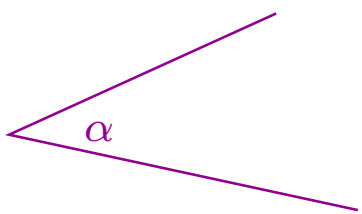


Construction 3; page 161 - Angle of 90 °

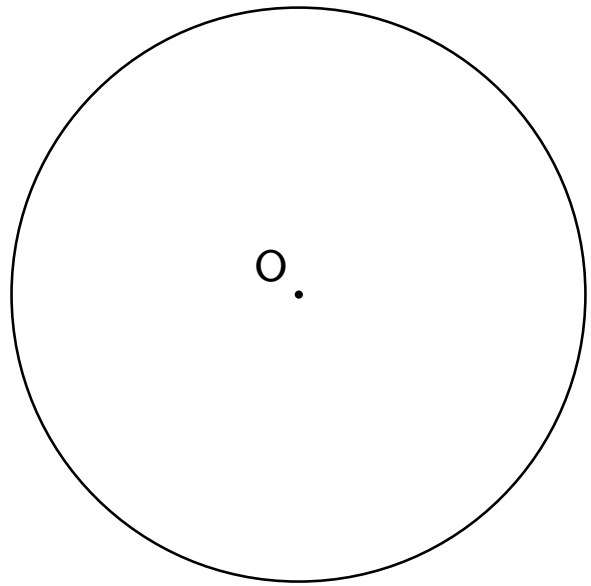


Construction 3; page 161 - Given an angle





An equilateral triangle



Sum of the angles of a triangle

In a hyperbolic plane, the sum of the angles of a triangle is LESS THAN 180 °.