UNIVERSITY OF MANITOBA

DATE: February 5, 2010

COURSE NO: MATH 2500
EXAMINATION:
Introduction to Number Theory

MIDTERM I

TIME: 50 minutes
EXAMINER: M. Davidson

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[10] 1. Use **mathematical induction** to show that $2+5+8+\ldots+(6n-1)=n(6n+1)$.

Let $P(n)$ be the statement

$$2+5+8+\cdots+(6n-1) = n(6n+1)$$

If $n=1$ Then $2+5=7$

and $1(6+1)=7$

Hence $P(1)$ is true.

Assume $P(k)$ is true:

$$2+5+8+\cdots+(6k-1) = k(6k+1).$$

Then $2+5+8+\cdots+(6(k+1)-1)$

$= 2+5+8+\cdots+(6k-1)+(6k+2)+(6k+5)$

$= k(6k+1)+6k+2+6k+5$

$= 6k^2+k+12k+7$

$= 6k^2+13k+7$

$= (k+1)(6k+7)$

$= (k+1)(6(k+1)+1)$

So $2+5+8+\cdots+(6(k+1)-1) = (k+1)(6(k+1)+1)$; ie $P(k+1)$ is true.

Since $P(1)$ is true and $P(k)$ implies $P(k+1)$, by PMI
$P(n)$ is true for all $n \geq 1$.

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[7] 2. (a) Show: If $a \equiv b \pmod m$ and $d \mid m$, then $a \equiv b \pmod d$. Justify each step.

Since $a \equiv b \mod m$, we have $m \mid a-b$.

since $m \mid a-b$, there is an integer r such that $a-b = rm$.

Since $d \mid m$, there is an integer S such that $dS = m$.

from above we get $a-b = dsr$; Since sr is an integer, $d \mid a-b$, hence $a \equiv b \mod d$.

(b) Using the above, show that the following system has no solution.

$x \equiv 47 \pmod{77}$
$x \equiv 14 \pmod{43}$
$x \equiv 74 \pmod{121}$

$x \equiv 47 \mod 77$ and $11 \mid 77$ so $x \equiv 47 \mod 11$ or $x \equiv 3 \mod 11$

$x \equiv 74 \mod 121$ and $11 \mid 121$ so $x \equiv 74 \mod 11$ or $x \equiv 8 \mod 11$

So there is no x that satisfies the above

[3] 3. For what primes m is the following congruence true :
$1815 \equiv 1542 \pmod m$

$m \mid 1815 - 1542$

so $m \mid 273$

The prime power decomposition of 273 is $3 \cdot 7 \cdot 13$

Hence m could be 3 or 7 or 13

[10] 4. (a) Use the Euclidean algorithm to find (7364, 553).

(b) Find all solutions to $7364x + 553y = 91$.

(c) Find all solutions to $553x \equiv 91 \pmod{7364}$

a) $7364 = 553(13) + 175$

$553 = 175(3) + 28$

$175 = 28(6) + 7$

$28 = 7(4) + 0$

Hence $(7364, 553) = 7$

b) $7 = 175 + 28(-6)$

$= 175 + [553 + 175(-3)](-6)$

$= 553(-6) + 175(19)$

$= 553(-6) + [7364 + 553(-13)](19)$

$= 7364(19) + 553(-253)$

So $7364(19) + 553(-253) = 7$

$7364(247) + 553(-3289) = 91$

All solutions to $7364x + 553y = 91$ are

$x = 247 + \frac{553}{7}t = 247 + 79t$

$y = -3289 - \frac{7364}{7}t = -3289 - 1052t$

c) The 7 solutions are

919, 1971, 3023, 4075, 5127, 6179, 7231

[10] 5. Write the following as a single congruence, if possible. If it is not possible, explain why not.

$x \equiv 2 \pmod{9}$
$x \equiv 7 \pmod{13}$
$x \equiv 13 \pmod{380}$

$x \equiv 2 \bmod 9$

So $x = 2 + 9k_1$

$2 + 9k_1 \equiv 7 \bmod 13$

$9k_1 \equiv 5 \bmod 13$

$9k_1 \equiv 18 \bmod 13$

$k_1 \equiv 2 \bmod 13$

$k_1 = 2 + 13k_2$

$x = 2 + 9(2 + 13k_2)$

$= 20 + 117k_2$

Alternately:

$13 = 9(1) + 4$
$9 = 4(2) + 1$

$1 = 9 + 4(-2)$
$= 9 + [13 + 9(-1)](-2)$
$= 13(-2) + 9(3)$

$9k_1 \equiv 5 \bmod 13$
$3 \cdot 9k_1 \equiv 3 \cdot 5 \bmod 13$
$k_1 \equiv 15 \bmod 13$
$k_1 \equiv 2 \bmod 13$

$20 + 117k_2 \equiv 13 \bmod 380$

$117k_2 \equiv -7 \bmod 380$

$k_2 \equiv -91 \bmod 380$

$k_2 \equiv 289 \bmod 380$

$k_2 = 289 + 380k_3$

$380 = 117(3) + 29$
$117 = 29(4) + 1$

$1 = 117 + 29(-4)$
$= 117 + [380 + 117(-3)](-4)$
$= 380(-4) + 117(13)$

$x = 20 + 117(289 + 380k_3)$

$= 33833 + 44460k_3$

The single congruence.

$$\boxed{x \equiv 33833 \bmod 44460}$$