THE UNIVERSITY OF MANITOBA

DATE: December 14, 2012

FINAL EXAMINATION

DEPARTMENT & NO: MATH2130

TIME: 3 hours

EXAMINATION: Engineering Mathematical Analysis 1

EXAMINER: M. Davidson, D. Trim

INSTRUCTIONS:

1. No aids permitted.
2. Attempt all questions.
3. If insufficient space is provided for a solution to a problem, continue your work on the back of the previous page.
4. Check that your examination booklet contains questions numbered from 1 to 12.
5. Fill in the information requested below.

Student Name (Print): Solutions

Student Signature:

Student Number:

Seat Number:

Circle your instructor's name: M. Davidson D. Trim

Question	Maximum Mark	Assigned Mark	Question	Maximum Mark	Assigned Mark
1	8		7	6	
2	6		8	7	
3	5		9	8	
4	9		10	8	
5	8		11	12	
6	14		12	9	
Total	50		Total	50	

Examination Total /100

8 1. Find parametric equations for the tangent line to the curve

$$x^2y + z^3 + xz = 9, \quad xy + y^4z = 1,$$

at the point $(1, -1, 2)$.

Let $F = x^2y + z^3 + xz - 9$

So $\nabla F = \langle 2xy + z, x^2, 3z^2 + x \rangle$

$\nabla F \big|_{(1,-1,2)} = \langle -2+2, 1, 12+1 \rangle = \langle 0, 1, 13 \rangle$

which is a vector perpendicular to the curve at the given point

Let $G = xy + y^4z - 1$

So $\nabla G = \langle y, x + 4y^3z, y^4 \rangle$

$\nabla G \big|_{(1,-1,2)} = \langle -1, 1 + (-8), 1 \rangle = \langle -1, -7, 1 \rangle$

another vector perpendicular to the curve at the given point.

So a vector tangent to the curve is

$$\vec{V} = \nabla F \times \nabla G = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 13 \\ -1 & -7 & 1 \end{vmatrix} = \langle 1 + 91, -13, 1 \rangle$$

$$= \langle 92, -13, 1 \rangle$$

So the parametric equation of the tangent line is

$x = 1 + 92t$

$y = -1 - 13t$

$z = 2 + t$

$(t \in \mathbb{R})$

6 **2.** Set up, but do **NOT** evaluate a definite integral for the length of the curve

$$z = x^2 + y^2, \quad 2x - 4y + z = 4.$$

You need not simplify the integral.

We parametrize the curve as follows:

$$z = x^2 + y^2 \qquad z = 4 - 2x + 4y$$

$$x^2 + y^2 = 4 - 2x + 4y$$

$$x^2 - 2x + y^2 - 4y = 4$$

$$x^2 - 2x + 1 + y^2 - 4y + 4 = 4 + 1 + 4$$

$$(x+1)^2 + (y-2)^2 = 9$$

$$\left(\frac{x+1}{3}\right)^2 + \left(\frac{y-2}{3}\right)^2 = 1$$

so let $\frac{x+1}{3} = \cos t$ and $\frac{y-2}{3} = \sin t$ $\quad (0 \le t \le 2\pi)$

$$x = 3\cos t - 1 \qquad y = 3\sin t + 2$$

and $z = 4 - 2x + 4y = 4 - 2(3\cos t - 1) + 4(3\sin t + 2)$

$$= 4 - 6\cos t + 2 + 12\sin t + 8$$

$$= 12\sin t - 6\cos t + 14$$

Now $\frac{dx}{dt} = -3\sin t$

$$\frac{dy}{dt} = 3\cos t$$

$$\frac{dz}{dt} = 12\cos t + 6\sin t$$

Substituting these values into the formula for length $\left(L = \int_\alpha^\beta \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}\, dt \right)$ gives

$$L = \int_0^{2\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (12\cos t + 6\sin t)^2}\, dt$$

5 3. Find the equation of the plane that passes through the point $(4, -3, 5)$ and is perpendicular to the line

$$\frac{2x-1}{10} = \frac{y+5}{-7} = \frac{1-z}{3}.$$

Simplify the equation as much as possible.

Rewriting the equation of the line into standard form, we find

$$\frac{x-\frac{1}{2}}{5} = \frac{y+5}{-7} = \frac{z-1}{-3}$$

we find the vector $\langle 5,-7,-3\rangle$ as the vector in the direction of the line. Since the line is perpendicular to the desired plane, the vector $\langle 5,-7,-3\rangle$ is also normal to the plane.

The equation of a plane with normal $\langle 5,-7,-3\rangle$ and passing through $(4,-3,5)$ is

$$5(x-4)-7(y+3)-3(z-5)=0$$

Simplifying, we find:

$$5x-20-7y-21-3z+15=0$$

$$5x-7y-3z=20+21-15$$

$$5x-7y-3z=26.$$

9 **4.** The equations

$$x = u^2 + v^3, \quad y = 3uv + u^2v^2$$

explicitly define x and y as functions of u and v. They also implicitly define u and v as functions of x and y. Show that

$$\left.\frac{\partial u}{\partial x}\right)_y \neq \frac{1}{\left.\frac{\partial x}{\partial u}\right)_v}.$$

Let $F = x - u^2 - v^3$

and $G = y - 3uv - u^2v^2$

$$\left.\frac{\partial u}{\partial x}\right)_y = \frac{-\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} = \frac{-\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = \frac{-\begin{vmatrix} 1 & -3v^2 \\ 0 & -3u-2u^2v \end{vmatrix}}{\begin{vmatrix} -2u & -3v^2 \\ -3v-2uv^2 & -3u-2u^2v \end{vmatrix}}$$

$$= \frac{-(-3u-2u^2v)}{2u(3u+2u^2v)-(9v^3+6uv^4)} = \frac{3u+2u^2v}{6u^2+4u^3v-9v^3-6uv^4}$$

Whereas

$$\left.\frac{\partial x}{\partial u}\right)_v = 2u \quad \text{so} \quad \frac{1}{\left.\frac{\partial x}{\partial u}\right)_v} = \frac{1}{2u}$$

8 **5.** Show that for any differentiable function f whatsoever, the function $u(x, y) = f(x^2 - 2y^2) + x^4$ satisfies the equation

$$2y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 8x^3y.$$

Let $s = x^2 - 2y^2$ and we see

$$\begin{tikzcd}[column sep=small] & & u \arrow[dl, dash] \arrow[dr, dash] & \\ & s \arrow[dl, dash] \arrow[dr, dash] & & x \\ x & & y & \end{tikzcd}$$

Now $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial s} \cdot \dfrac{\partial s}{\partial x} + \left.\dfrac{\partial u}{\partial x}\right)_s$

$= f'(s) \cdot 2x + 4x^3$

and $\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial s} \cdot \dfrac{\partial s}{\partial y} = f'(s) \cdot (-4y)$

So now

$$2y\left(\frac{\partial u}{\partial x}\right) + x\left(\frac{\partial u}{\partial y}\right)$$

$$= 2y\left(f'(s) \cdot 2x + 4x^3\right) + x\left(f'(s)(-4y)\right)$$

$$= 4xy \cdot f'(s) + 8x^3y - 4xy\, f'(s)$$

$$= 8x^3y$$

which verifies the above equation.

14 6. Find the maximum and minimum values of the function

$$f(x, y) = xy(1 - 2x - 2y)$$

on the region consisting of the triangle enclosed by the lines

$$x + y = 1, \quad x = 0, \quad y = 0$$

and its edges.

$f(x,y) = xy(1-2x-2y) = xy - 2x^2y - 2xy^2$

To find the critical points:

$f_x = y - 4xy - 2y^2 = y(1-4x-2y)$ and

$f_y = x - 2x^2 - 4xy = x(1-2x-4y)$

So $f_x = 0$ if $y=0$ OR $1-4x-2y=0$

$f_y = 0$ if $x=0$ OR $1-2x-4y=0$

We find the 4 critical Points:

$(y=0, x=0)$ $(0,0)$

$(y=0, 1-2x-4y=0)$ $(\frac{1}{2}, 0)$

$(1-4x-2y=0, x=0)$ $(0, \frac{1}{2})$

We note that these C.P.'s are on the boundary, or we check values

$f(0,0)=0$

$f(\frac{1}{2},0)=0$

$f(0,\frac{1}{2})=0$

$(1-4x-2y=0, 1-2x-4y=0)$ $(\frac{1}{6}, \frac{1}{6})$

$f(\frac{1}{6}, \frac{1}{6}) = (\frac{1}{6})(\frac{1}{6})[1-\frac{2}{6}-\frac{2}{6}]$
$= \frac{1}{36}(\frac{2}{6}) = \frac{1}{108}$

We now check the boundaries:

If $x=0$, then $f(0,y)=0$ (so both max & min value on that boundary is 0)

If $y=0$, then $f(x,0)=0$

If $x+y=1$, then $y=1-x$ (on $[0,1]$) and

$g(x) = f(x,1-x) = x(1-x)(1-2x-2(1-x)) = x(1-x)(-1) = x^2-x$

So $g'(x) = 2x-1$ and $g'(x)=0 \Rightarrow 2x-1=0 \Rightarrow x = 1/2$

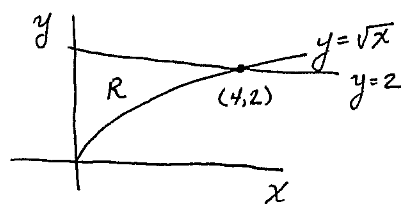
$g(\frac{1}{2}) = (\frac{1}{2})^2 - \frac{1}{2} = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$ and $g(0)=g(1)=0$

Hence the maximum value of the function on this region is $\frac{1}{108}$, and the minimum value is $-1/4$.

6 **7.** Evaluate the double iterated integral

$$\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\, dy\, dx.$$

We first change the order of integration:



So $\int_0^4 \int_{\sqrt{x}}^2 e^{y^3}\, dy\, dx$

$$= \int_0^2 \int_0^{y^2} e^{y^3}\, dx\, dy$$

$$= \int_0^2 x e^{y^3} \Big]_0^{y^2} dy$$

$$= \int_0^2 y^2 e^{y^3}\, dy$$

$$= \frac{1}{3} \int_0^2 3y^2 e^{y^3}\, dy$$

$$= \frac{1}{3} \left(e^{y^3} \Big]_0^2 \right.$$

$$= \frac{1}{3} \left(e^8 - e^0 \right)$$

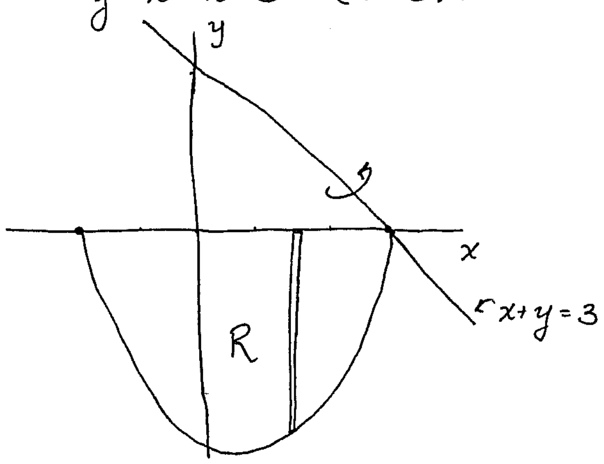
$$= \frac{1}{3} \left(e^8 - 1 \right)$$

7 **8.** Set up, but do **NOT** evaluate a double iterated integral for the volume of the solid of revolution when the area bounded by the curves

$$y = x^2 - x - 6, \quad y = 0,$$

is rotated about the line $x + y = 3$. Simplify the integrand as much as possible.

$y = x^2 - x - 6 = (x-3)(x+2)$



We note that the distance between a point (x, y) and the line $x+y=3$ is

$$d = \frac{|x+y-3|}{\sqrt{2}}$$

Testing the point $(0,0)$, which is on the same side of $x+y=3$ as all pts in R, we get $d = \frac{|-3|}{\sqrt{2}} = \frac{3}{\sqrt{2}}$, note we take the negative of $x+y-3$.

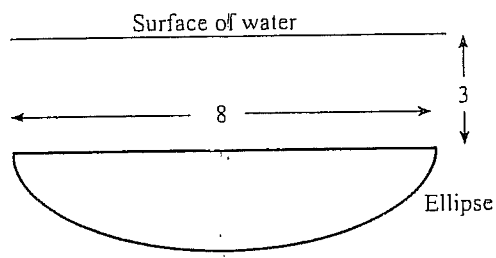
So $d = \dfrac{-x-y+3}{\sqrt{2}}$ for all points in R.

So

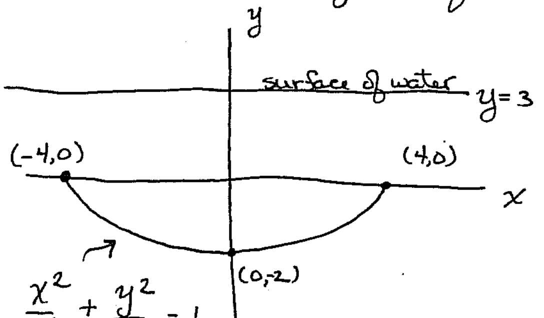
$$\text{Volume} = \iint_R 2\pi r \, dA$$

$$= \int_{-2}^{3} \int_{x^2-x-6}^{0} 2\pi \left(\frac{-x-y+3}{\sqrt{2}} \right) dy \, dx$$

8 9. Pictured below is a semi-elliptic plate submerged vertically in water. All dimensions are in metres. Set up, but do NOT evaluate, a double iterated integral for the force on each side of the plate due to the water.



We coordinatize as follows



$$\frac{x^2}{4^2}+\frac{y^2}{2^2}=1$$

or

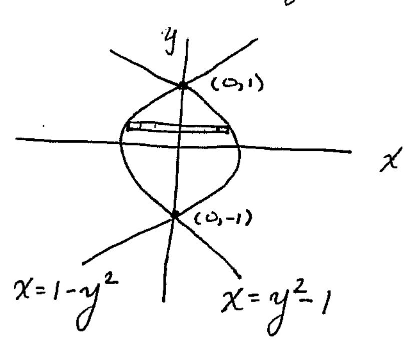
$$y=-\sqrt{2^2-\frac{2^2x^2}{4^2}}=-\sqrt{4-\frac{x^2}{4}}=-\sqrt{\frac{16-x^2}{4}}$$

So Force $= \iint_R g\rho d \, dA$ N

$$= \int_{-4}^{4}\int_{-\sqrt{\frac{16-x^2}{4}}}^{0} (9.81)(1000)(3-y)\,dx\,dy \text{ N}$$

8 10. Set up, but do **NOT** evaluate, a double iterated integral for the surface area of that portion of the surface $ze^{2x+3y} = 1$ that is enclosed by the surfaces $x = 1 - y^2$ and $x = y^2 - 1$.

We note that the projection of the region is sketched as follows



Now considering the surface $ze^{2x+3y} = 1$

we find $z = e^{-2x-3y}$

so $\frac{\partial z}{\partial x} = (-2)e^{-2x-3y}$

and $\frac{\partial z}{\partial y} = (-3)e^{-2x-3y}$

and hence $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+(-2e^{-2x-3y})^2+(-3e^{-2x-3y})^2}$

$= \sqrt{1+4e^{-4x-6y}+9e^{-4x-6y}} = \sqrt{1+13e^{-4x-6y}}$

Hence Surface Area $= \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA$

$$= \int_{-1}^{1}\int_{y^2-1}^{1-y^2} \sqrt{1+13e^{-4x-6y}}\; dx\, dy$$

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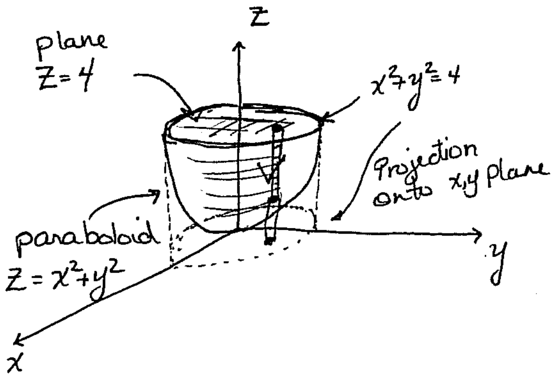
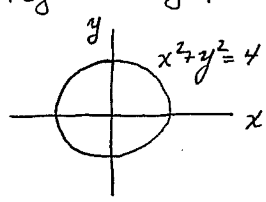
12 11. Evaluate the triple integral of the function $f(x, y, z) = 2(x^2 + y^2)$ over the volume bounded by the surfaces

$$z = x^2 + y^2, \quad z = 4.$$



Region in x,y-plane



$x^2+y^2 \le z \le 4$

In Cylindrical coordinates we find $z = x^2+y^2$ becomes $z = r^2$

So $\iiint_V 2(x^2+y^2)\,dV$

$$= \int_0^2 \int_0^{2\pi} \int_{r^2}^4 2r^2\, r\, dz\, d\theta\, dr = \int_0^2 \int_0^{2\pi} \int_{r^2}^4 2r^3\, dz\, d\theta\, dr$$

$$= \int_0^2 \int_0^{2\pi} 2r^3 z \Big]_{r^2}^4 \, d\theta\, dr$$

$$= \int_0^2 \int_0^{2\pi} 8r^3 - 2r^5 \, d\theta\, dr$$

$$= \int_0^2 (8r^3 - 2r^5)\theta \Big]_0^{2\pi} dr$$

$$= \int_0^2 16\pi r^3 - 4\pi r^5 \, dr$$

$$= \left(4\pi r^4 - \frac{4}{6}\pi r^6\right]_0^2 = 64\pi - \frac{2}{3}(64\pi)$$

$$= \frac{192\pi - 128\pi}{3} = \frac{64\pi}{3}$$

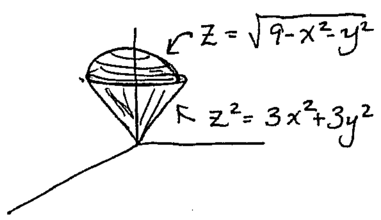
9 **12.** Set up, but do **NOT** evaluate, a triple iterated integral in spherical coordinates for the triple integral

$$\iiint_V (x^2 + y^2)\, dV$$

where V is the region bounded by the surfaces

$$z^2 = 3x^2 + 3y^2, \quad z = \sqrt{9 - x^2 - y^2}.$$

We note that $z^2 = 3x^2 + 3y^2$ is a cone, and $z = \sqrt{9-x^2-y^2}$ is the upper portion of a sphere. A rough sketch of the volume is:



We transform using

$x = R \sin\phi \cos\theta$

$y = R \sin\phi \sin\theta$

$z = R \cos\phi$

$(dV = R^2 \sin\phi \, dR\, d\theta\, d\phi)$

From this, we see that R is bounded by the sphere $z = \sqrt{9-x^2-y^2}$

or $z^2 + x^2 + y^2 = 9$

so $R^2 = 9$ and $R = 3$ (R is positive).

The angle ϕ is bounded by the cone $z^2 = 3x^2 + 3y^2$

so $R^2\cos^2\phi = 3R^2\sin^2\phi\cos^2\theta + 3R^2\sin^2\phi\sin^2\phi$

$R^2\cos^2\phi = 3R^2\sin^2\phi$

$\cos^2\phi = 3\sin^2\phi$

$1 - \sin^2\phi = 3\sin^2\phi$

$4\sin^2\phi = 1$

$\sin^2\phi = 1/4$

$\sin\phi = 1/2$ (we see from the sketch the sine should be positive)

so $\phi = \pi/6$

And we transform the function x^2+y^2 into $R^2\sin^2\phi\cos^2\theta + R^2\sin^2\phi\sin^2\theta$

$= R^2\sin^2\phi$.

So $\iiint_V (x^2+y^2)\, dV$

$$= \int_0^{\pi/6}\int_0^{2\pi}\int_0^{3} R^4 \sin^3\phi \, dR\, d\theta\, d\phi$$