Math 2500 W2013 Assignment 1 - Solutions

Q1. Let $P(n)$ be the statement

$$3+7+11+15+\cdots+(12n-1) = 3n(6n+1)$$

If $n=1$, we note that $12(1)-1 = 11$ so

$$3+7+11 = 21$$

and $3(6+1) = 3\cdot 7 = 21$

So $P(1)$ is true.

Suppose $P(k)$ was true; ie

$$3+7+11+15+\cdots+(12k-1) = 3k(6k+1)$$

Then $3+7+11+15+\cdots+(12(k+1)-1)$

$$\begin{aligned}
&= 3+7+11+15+\cdots+(12k+11) \\
&= 3+7+11+15+\cdots+(12k-1)+(12k+3)+(12k+7)+(12k+11) \\
&= 3k(6k+1) + 36k + 21 \quad \text{by ind. hyp.} \\
&= 18k^2 + 3k + 36k + 21 \\
&= 18k^2 + 39k + 21 \\
&= (3k+3)(6k+7) \\
&= 3(k+1)(6(k+1)+1)
\end{aligned}$$

So $P(k+1)$ is also true.

Since $P(1)$ is true and $P(k)$ implies $P(k+1)$, by PMI, $P(n)$ is true for all $n \geq 1$.

Q2. Let $P(n)$ be the statement

$$(2n+1)+(2n+3)+(2n+5)+\cdots+(4n+1)=(3n+1)(n+1)$$

When $n=1$ $\quad 2n+1=3$ and $4n+1=5$

So $\quad 3+5=8$

and $\quad (3+1)(1+1)=4\cdot 2=8$

So $P(1)$ is true.

Suppose $P(k)$ is true, so

$$(2k+1)+(2k+3)+(2k+5)+\cdots+(4k+1)=(3k+1)(k+1)$$

Then $(2(k+1)+1)+(2(k+1)+3)+(2(k+1)+5)+\cdots+(4(k+1)+1)$

$= (2k+3)+(2k+5)+(2k+7)+\cdots+(4k+5)$

$= (2k+3)+(2k+5)+(2k+7)+\cdots+(4k+1)+(4k+3)+(4k+5)$

$= (2k+1)+(2k+3)+(2k+5)+\cdots+(4k+1)+(4k+3)+(4k+5)-(2k+1)$ by ind. hyp.

$= (3k+1)(k+1)+6k+7$

$= 3k^2+4k+1+6k+7$

$= 3k^2+10k+8$

$= (3k+4)(k+2)$

$= (3(k+1)+1)((k+1)+1)$

So $P(k+1)$ is also true.

Since $P(1)$ is true and $P(k)$ implies $P(k+1)$, by PMI, $P(n)$ is true for all $n \geq 1$.

3. To find $(10017, 4480)$ we use the following:

$$10017 = 4480(2) + 1057$$
$$4480 = 1057(4) + 252$$
$$1057 = 252(4) + 49$$
$$252 = 49(5) + 7$$
$$49 = 7(7) + 0$$

So $(10017, 4480) = 7$

To solve $10017x + 4480y = 7$ we take the steps backwards:

$$\begin{aligned}
7 &= 252 + 49(-5) \\
&= 252 + [1057 + 252(-4)](-5) \\
&= 1057(-5) + 252(21) \\
&= 1057(-5) + [4480 + 1057(-4)](21) \\
&= 4480(21) + 1057(-89) \\
&= 4480(21) + [10017 + 4480(-2)](-89) \\
&= 10017(-89) + 4480(199)
\end{aligned}$$

So $10017(-89) + 4480(199) = 7$

[Hence $x = -89$ and $y = 199$ is a solution]

4 (a) i. $110880 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$

ii. $149600 = 2^5 \cdot 5^2 \cdot 11 \cdot 17$

iii. $1126125 = 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13$

(b) i. $(110880, 149600)$

$= (2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 17^0, \ 2^5 \cdot 3^0 \cdot 5^2 \cdot 7^0 \cdot 11 \cdot 17)$

$= 2^5 \cdot 3^0 \cdot 5 \cdot 7^0 \cdot 11 \cdot 17^0 = 1760$

ii. $(110880, 1126125)$

$= (2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13^0, \ 2^0 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13)$

$= 2^0 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13^0 = 3465$

iii. $(149600, 1126125)$

$= (2^5 \cdot 3^0 \cdot 5^2 \cdot 7^0 \cdot 11 \cdot 13^0 \cdot 17, \ 2^0 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 17^0)$

$= 2^0 \cdot 3^0 \cdot 5^2 \cdot 7^0 \cdot 11 \cdot 13^0 \cdot 17^0 = 275$

5 (a) Since $(c,a)=d$, there are integers x & y such that $cx+ay=d$. (Theorem 1.4)

So $cbx+aby=db$

Since c/ab $c/cbx+aby$ (Lemma 1.2)
so c/db.

(b) Since $d|(a+b)$ and $d/(a-b)$, then
$d|(a+b)+(a-b)$; or $d|2a$. (Lemma 1.2)
Since d is odd, $(d,2)=1$ so $d|a$. (Corollary 1.1)

Similarly, $d|(a+b)-(a-b)$; ie. $d|2b$
and hence $d|b$.

Now, since $d|a$ and $d|b$
$d|(a,b)$ (Corollary 1.2)