Math 2500 Assignment 3 Winter 2013

1(a) Let $x = 1152 \cdot 1151 \cdots 108 \cdot 105 \cdot 104 \cdots 3 \cdot 2 \cdot 1$

Then $107 \cdot 106\, x = 1152!$

Since 1153 is prime, by Wilson's Theorem

$$1152! \equiv -1 \mod 1153$$

$$107 \cdot 106\, x \equiv -1 \mod 1153$$

$$11342\, x \equiv -1 \mod 1153$$

$$965 x \equiv -1 \mod 1153$$

(Reversing)

$1153 = 965(1) + 188$	$1 = 13 + 12(-1)$
$965 = 188(5) + 25$	$= 25(-1) + 13(2)$
$188 = 25(7) + 13$	$= 188(2) + 25(-15)$
$25 = 13(1) + 12$	$= 965(-15) + 188(77)$
$13 = 12(1) + 1$	$= 1153(77) + 965(-92)$

So $x \equiv 92 \mod 1153$, 92 is the least residue.

(b) $1147 = 31 \times 37$ and $\phi(1147) = 30 \times 36 = 1080$

hence $13^{1080} \equiv 1 \mod 1147$, by Euler's Theorem

now $13^{56162} \equiv 13^{1080(52)+2} \equiv (13^{1080})^{52} \cdot 13^{2} \equiv 169 \mod 1147$.

The least residue is 169.

(c) Let $X = 1072 \cdot 1071 \cdots 532 \cdot 530 \cdots 3 \cdot 2 \cdot 1$

Since $1073 = 29 \times 37$, we note that $1073 \mid x$,

hence $x \equiv 0 \mod 1073$.

The least residue is 0.

(d) Since 1171 is prime, $6^{1170} \equiv 1 \mod 1171$, by Fermat's Theorem.

So $6^{49144} \equiv 6^{1170(42)+4} \equiv (6^{1170})^{42} \cdot 6^4 \equiv 1296 \equiv 125 \mod 1171$

The least residue is 125.

(e) Since 1283 is prime, $5^{1282} \equiv 1 \mod 1283$, by Fermat's Theorem.

So $5^{50008} \equiv 5^{1282(39)+10} \equiv (5^{1282})^{39} \cdot 5^{10} \equiv 9765625 \equiv 712 \mod 1283$

The least residue is 712.

(f) Let $x = 1068 \cdot 1067 \cdot 1066 \cdots 7 \cdot 6 \cdot 5$

So $4 \cdot 3 \cdot 2\, x = 1068!$

Since 1069 is prime $1068! \equiv -1 \mod 1069$

So $24x \equiv -1 \mod 1069$

	(Reversing)
$1069 = 24(44) + 13$	$1 = 11 + 2(-5)$
$24 = 13(1) + 11$	$= 13(-5) + 11(6)$
$13 = 11(1) + 2$	$= 24(6) + 13(-11)$
$11 = 2(5) + 1$	$= 1069(-11) + 24(490)$

So $x \equiv -490 \equiv 579 \mod 1069$, 579 is the least residue

(g) $1121 = 19 \times 59$ and $\phi(1121) = 18 \times 58 = 1044$

So $7^{1044} \equiv 1 \mod 1121$, by Euler's Theorem.

Now $7^{61603} \equiv 7^{1044(59)+7} \equiv (7^{1044})^{59} \cdot 7^7 \equiv 823543 \equiv 729 \mod 1121$

So 729 is the least residue

2.

(a) $205821 = 3^5 \cdot 7 \cdot 11^2$

$d(205821) = 6 \cdot 2 \cdot 3 = 36$

$\sigma(205821) = \left(3^6-1/3-1\right)(8)\left(11^3-1/11-1\right) = 364 \cdot 8 \cdot 133$
$= 387296$

$\phi(205821) = 3^4(3-1)(7-1)11(11-1) = 3^4 \cdot 2 \cdot 6 \cdot 11 \cdot 10$
$= 106920$

(b) $29766 = 2 \cdot 3 \cdot 11^2 \cdot 41$

$d(29766) = 2 \cdot 2 \cdot 3 \cdot 2 = 24$

$\sigma(29766) = 3 \cdot 4 \cdot 133 \cdot 42 = 67032$

$\phi(29766) = 1 \cdot 2 \cdot 11 \cdot 10 \cdot 40 = 8800$

(c) $3577392 = 2^4 \cdot 3^3 \cdot 7^2 \cdot 13^2$

$d(3577392) = 5 \cdot 4 \cdot 3 \cdot 3 = 180$

$\sigma(3577392) = (2^5-1)\left(\frac{3^4-1}{3-1}\right)\left(\frac{7^3-1}{7-1}\right)\left(\frac{13^3-1}{13-1}\right) = 31 \cdot 40 \cdot 57 \cdot 183$
$= 12934440$

$\phi(3577392) = 2^3 \cdot 3^2(3-1) \cdot 7(7-1) \cdot 13(13-1) = 2^3 3^2 \cdot 2 \cdot 7 \cdot 6 \cdot 13 \cdot 12$
$= 943488.$

3(a) $33550336 = 2^{12} \cdot 8191$

So $\sigma(33550336) = (2^{13}-1)(8192)$
$= 67100672$
$= 2(33550336)$

Hence 33550336 is perfect.

(b) $523776 = 2^9 \cdot 3 \cdot 11 \cdot 31$

So $\sigma(523776) = (2^{10}-1) \cdot 4 \cdot 12 \cdot 32$
$= 1571328$
$= 3(523776)$

Hence 523776 is 3-perfect.

(c) $5020 = 2^2 \cdot 5 \cdot 251$
$\sigma(5020) = 7 \cdot 6 \cdot 252 = 10584$
$\sigma(5020) - 5020 = 5564$

$5564 = 2^2 \cdot 13 \cdot 107$
$\sigma(5564) = 7 \cdot 14 \cdot 108 = 10584$
$\sigma(5564) - 5564 = 5020$

Hence 5020 and 5564 are an amicable pair.

(d)

$12496 = 2^4 \cdot 11 \cdot 71$

$\sigma(12496) = (2^5-1)(12)(72) = 26784$

$\sigma(12496) - 12496 = 14288$

$14288 = 2^4 \cdot 19 \cdot 47$

$\sigma(14288) = (2^5-1)(20)(48) = 29760$

$\sigma(14288) - 14288 = 15472$

$15472 = 2^4 \cdot 967$

$\sigma(15472) = (2^5-1)(968) = 30008$

$\sigma(15472) - 15472 = 14536$

$14536 = 2^3 \cdot 23 \cdot 79$

$\sigma(14536) = (2^4-1)(24)(80) = 28800$

$\sigma(14536) - 14536 = 14264$

$14264 = 2^3 \cdot 1783$

$\sigma(14264) = (2^4-1)(1784) = 26760$

$\sigma(14264) - 14264 = 12496$

Showing that the aliquot sequence of 12496 repeats with a period of 5.

4. Since n is 4-perfect, $\sigma(n) = 4n$.
Since $(40, n) = 1$ and σ is multiplicative,

$$\begin{aligned}\sigma(40n) &= \sigma(40)\,\sigma(n)\\ &= \sigma(2^3)\,\sigma(5)\,\sigma(n)\\ &= 15 \cdot 6 \cdot 4n\\ &= 360n\\ &= 9(40n)\end{aligned}$$

Hence $40n$ would be 9-perfect.

5. Suppose $(r, n) = 1$, and let $(n-r, r) = d$.
so $d \mid r$ and $d \mid n-r$, so $d \mid (n-r)+r$ and so $d \mid n$. Since $d \mid r$ and $d \mid n$, $d \mid (r,n)$, or $d \mid 1$. Since $d > 0$ (d is a greatest common divisor) and $d \mid 1$ we get that $d = 1$.

Hence, if r is relatively prime to n, then $n-r$ is also relatively prime to n. If r and $n-r$ were not distinct, then
$r = n-r \Rightarrow n = 2r$ and the only way r would be relatively prime to n is if $r = 1$ $(n=2)$

If $n \neq 2$, then the least residues relatively prime to n can be paired up into $(\phi(n)/2)$ pairs summing to n, Hence the total sum is $\frac{\phi(n) \cdot n}{2}$.

If $n = 2$ then 1 is the only relatively prime positive integer, and $\frac{\phi(2) \cdot 2}{2} = 1$.