THE UNIVERSITY OF MANITOBA

August 22, 2003 — SUMMER DAY FINAL EXAMINATION

PAPER # 54 — PAGE NO. 1 of 2

DEPARTMENT & COURSE NO.: 136.250 — TIME: 2 Hours

EXAMINATION: Introduction to Number Theory — EXAMINER: M. Davidson

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INSTRUCTIONS:
Calculators are permitted. Answer all questions in the booklets provided. Show your work. You may answer the questions in any order, but clearly indicate which question you are answering. Put your name and student number on all booklets you hand in. In answering proof/show question, be sure to clearly justify all steps.

Primes you may want to know 2, 3, 5, 7, 11, 13, 17, 19, 31, 53, 311, 601, 739, 971, 2143

Values

[18] 1. Recall d(n) is the number of divisors of n, $\sigma(n)$ is the sum of divisors of n and $\phi(n)$ is the Euler phi function.

(a) What is $d(43560)$? $\sigma(43560)$? $\phi(43560)$? $\left(43560 = 2^3 \cdot 3^2 \cdot 5 \cdot 11^2\right)$.

(b) If p and q are prime, what is $d(p^3q)$? $\sigma(p^3q)$? $\phi(p^3q)$?

(c) Show that if n is a square, then $\sigma(n)$ is odd.

[10] 2. Find all solutions to $318x + 1668y = 42$.

[10] 3. Find all solutions to $280x \equiv 42 \mod 861$.

[12] 4. By induction, prove $1 \cdot 3 + 2 \cdot 4 + \cdots + (n-1)(n+1) = \dfrac{(n-1)(n)(2n+5)}{6}$ (for $n \geq 2$).

[14] 5. Write as a single congruence:

$x \equiv 2 \pmod 7$

$x \equiv 10 \pmod{23}$

$x \equiv 3 \pmod{53}$

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[10] 6. Twin primes are a pair of numbers that differ by 2 and are both prime (i.e. p and $p+2$ are both prime). Show that the number in between twin primes (except for the pair 3,5) is divisible by 6.
(Hint: Show every prime except 2 and 3 must be congruent to either 1 or 5 mod 6).

[8] 7. What is the least residue of 2140! mod 2143.
(Hint: How does this differ from the expression in Wilsons theorem?)

[18] 8. Calculate the following Legendre Symbols.

(a) $\left(\frac{311}{971}\right)$

(b) $\left(\frac{572}{601}\right)$

(c) $\left(\frac{456}{739}\right)$

[20] 9.
(a) How many primitive roots does 53 have?
(b) What possible orders could a non zero least residue of 53 have module 53.
(c) Show that 5 is a primitive root of 53.
(d) If $5^5 \equiv 51 \mod 53$, what is the order of 51 mod 53?
(e) If $5^6 \equiv 43 \mod 53$, what is the order of 43 mod 53?

[10+4]10.
(a) Define perfect, deficient and abundant.
(b) Show 496 is perfect.
(c) What are the divisors of an even perfect number?
(d) (bonus) Show that a divisor of an even perfect number, other than itself, is deficient.

[130]