UNIVERSITY OF MANITOBA

DATE: March 11, 2015

MIDTERM II
TITLE PAGE

COURSE: MATH 2400
EXAMINATION: Graph Theory

TIME: 60 minutes
EXAMINER: M. Davidson

FAMILY NAME: (Print in ink) SOLUTIONS

GIVEN NAME(S): (Print in ink) ______

STUDENT NUMBER: ______

SIGNATURE: (in ink) ______
(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 60 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 6 pages of questions. Please check that you have all the pages.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 60 points.

Answer questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	9	
2	10	
3	12	
4	10	
5	9	
6	10	
Total:	60	

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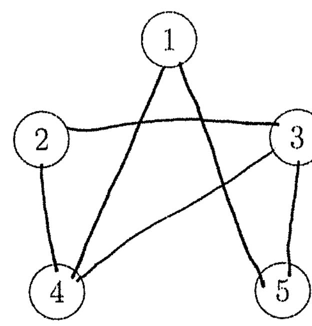
[9] 1. For the following questions, you will use the matrices:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 2 & 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 3 & 2 \\ 0 & 1 & 0 & 2 & 2 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 2 & 1 & 5 & 4 \\ 2 & 2 & 4 & 4 & 2 \\ 1 & 4 & 2 & 6 & 5 \\ 5 & 4 & 6 & 2 & 1 \\ 4 & 2 & 5 & 1 & 0 \end{pmatrix}.$$

(a) On the following vertices, draw the edges of the graph G that has A as its adjacency matrix.



(b) What are the features of the adjacency matrix A that tells you that the graph drawn in part(a) is a simple graph.

No loops: The diagonal is zero

No multiple edges: It is a $\{0,1\}$ matrix.

(There are no entries larger than 1).

(c) i. How many walks are there of length 3 from vertex 2 to vertex 5?

There are 2 walks

ii. How many walks are there of length 2 from vertex 3 to vertex 4?

There is 1 walk

(d) Find an incident matrix for the graph in part(a).

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[10] 2. *There are three schools in a town that compete for a sport cup. The school that has the cup will challenge one of the other schools to a game. The school that wins the game will then get (or keep) the cup. Every week a challenge is made, followed by a game. Over time, the following pattern has developed.*

If School A has the cup, then it will always challenge School B.
If School B has the cup, then it will challenge School C one quarter $\frac{1}{4}$ of the time.
If School C has the cup, then it will challenge School A half $\frac{1}{2}$ of the time.

When School A and School B play, School B wins two thirds $\frac{2}{3}$ of the time.
When School A and School C play, School C wins two thirds $\frac{2}{3}$ of the time.
When School C and School B play, School B wins two thirds $\frac{2}{3}$ of the time.

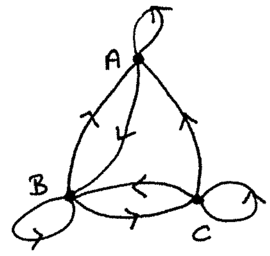
We consider a state to be based on what school currently has the cup.

(a) Find the Markov chain given that the cup starts at School C.

$$T = \begin{bmatrix} 1/3 & 2/3 & 0 \\ 1/4 & 2/3 & 1/12 \\ 1/6 & 1/3 & 1/2 \end{bmatrix}$$

$$\vec{x} = [0\ 0\ 1]$$

(b) Draw the digraph and find the associated adjacency matrix.



$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(c) Is the Markov chain from part (a) irreducible? (Explain)

Since the digraph in part (b) is strongly connected, the Markov chain is irreducible.

[12] 3. (a) State the definition of a tree.

a tree is a graph that is connected and contains no cycles.

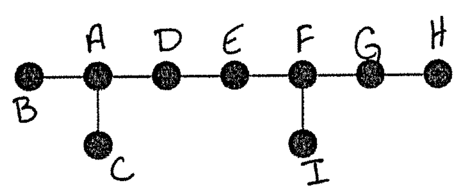
(b) State two conditions equivalent to being a tree.

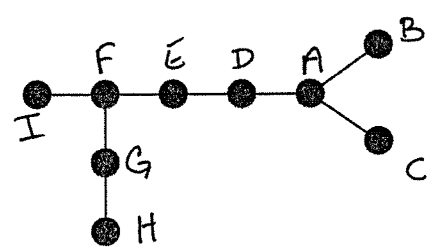
- a connected graph with $n-1$ edges.
- a graph with no cycles having $n-1$ edges.
- a connected graph where the removal of any edge disconnects the graph.
- a graph that has no cycles but the addition of any edge creates a cycle.
- a graph where there is exactly one path between every pair of vertices.

(c) Consider the following pairs of trees. If they are isomorphic, label the vertices to show they are. If they are not isomorphic, give a reason of why they are not.

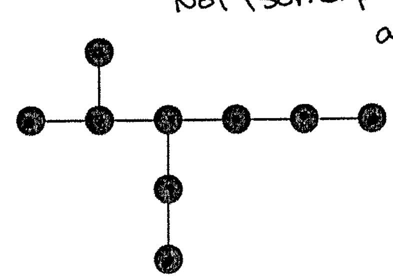
Isomorphic

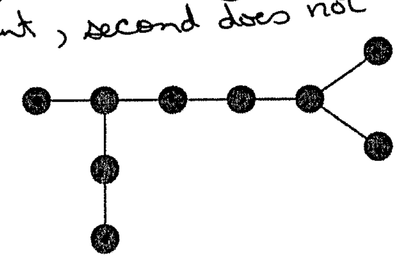
i.





ii. Not isomorphic, first has deg 3 vertices adjacent, second does not



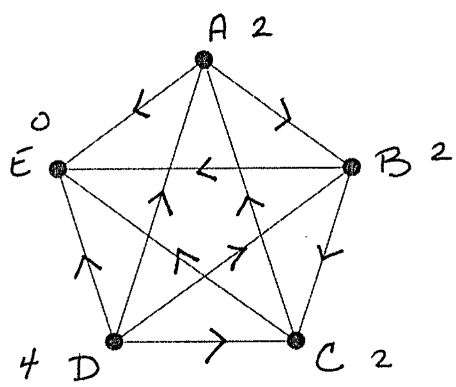


{1,1,1,1,2,2,2,3,3}

{1,1,1,1,2,2,2,3,3}

[10] 4. (a) Consider the following results of a survey of assorted snacks where the preferred choice is underlined. Draw the directed graph for the survey.

<u>Apple pie</u> - Bagel
<u>Bagel</u> - Cookies
Cookies - <u>Donut</u>
<u>Donut</u> - Eclair
Eclair - <u>Apple Pie</u>
Apple Pie - <u>Donut</u>
<u>Donut</u> - Bagel
<u>Bagel</u> - Eclair
Eclair - <u>Cookies</u>
<u>Cookies</u> - Apple Pie



(b) Define a tournament explain why this survey is a tournament.

A tournament is a digraph whose underlying graph is a complete graph.
The underlying graph for this survey is K_5, hence it is a tournament

(c) What is the score sequence?

$\{0, 2, 2, 2, 4\}$

(d) Is it transitive? (Explain.)

No, if it were then it would have $\{0,1,2,3,4\}$ as its score squence

(e) Is it strongly connected? (Explain.)

No, because it has a source. (It also has a sink.) Vertex D is a source. (Vertex E is a sink).

(f) If possible, find a ranking of the preferences.

Rankings:

D · A · B · C · E

D · B · C · A · E

D · C · A · B · E

[9] 5. For each of the following, draw a digraph (or digraphs) that satisfies the given properties if they exist. If no such digraph(s) exists, clearly show why not.

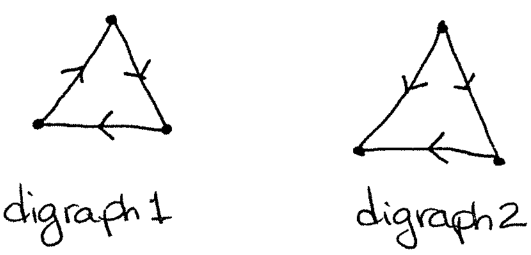
(a) A tournament with indegree sequence $\{1, 1, 2, 2, 3\}$

This does not exist.

A tournament on 5 vertices would have 10 arcs.

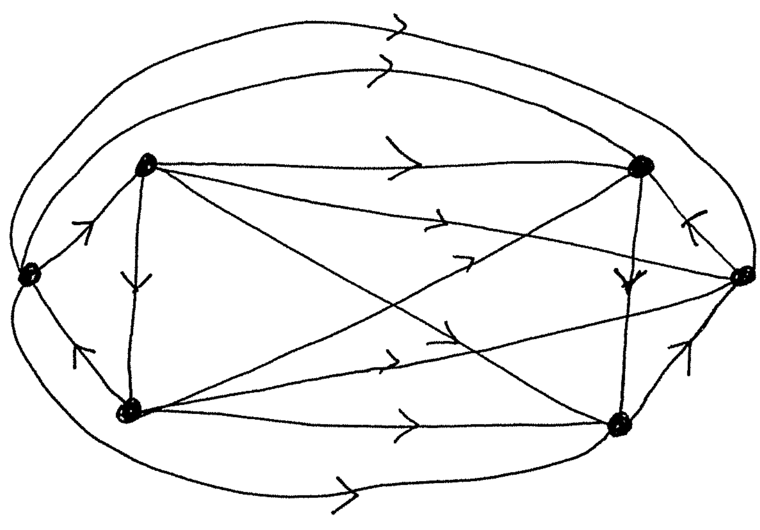
A digraph with indegree sequence $\{1,1,2,2,3\}$ would have $1+1+2+2+3=9$ arcs.

(b) A pair of digraphs that have the same underlying graph but the digraphs are not isomorphic.

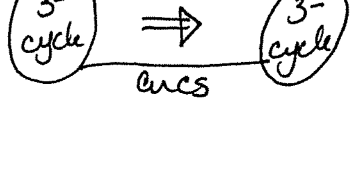


There are LOTS of examples satisfying this.

(c) A digraph whose underlying graph is K_6 that is not strongly connected, but does not have a source and does not have a sink.

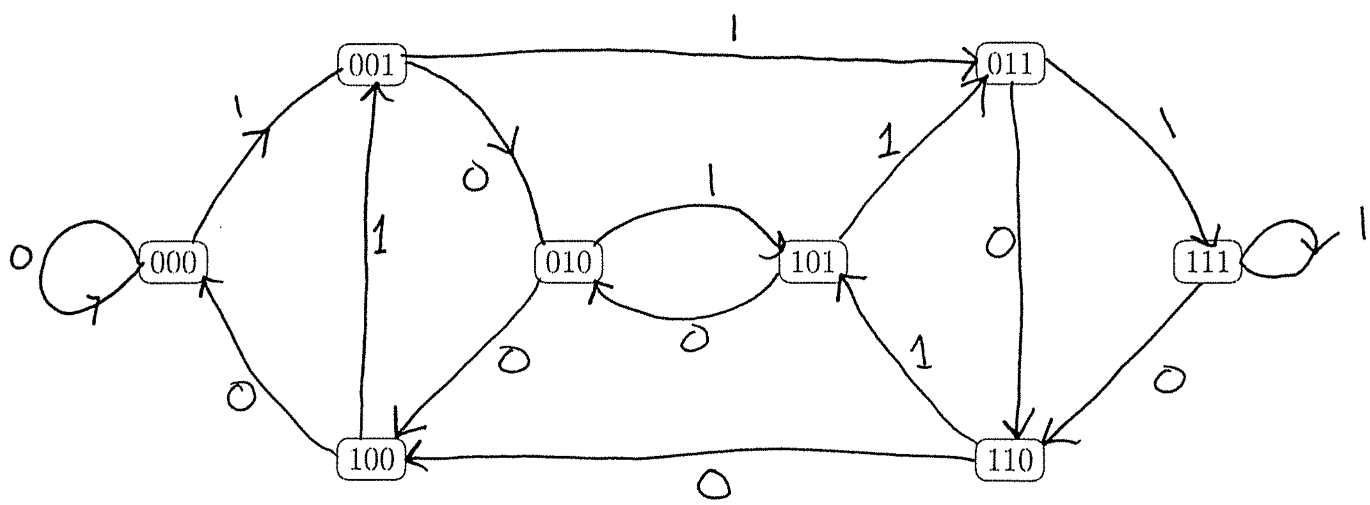


Pattern is



[10] 6. Solve the rotating drum problem:

- Draw the appropriate digraph on the labeled vertices provided.
- List the walk needed to solve the drum problem.
- Put 0's and 1's in the drum below so that every binary word of length 4 can be read as 4 consecutive bits.



WALK (NOT UNIQUE):

$000 \overset{0}{-} 000 \overset{1}{-} 001 \overset{0}{-} 010 \overset{1}{-} 101 \overset{0}{-} 010 \overset{0}{-} 100 \overset{1}{-} 001$
$\overset{1}{-} 011 \overset{1}{-} 111 \overset{1}{-} 111 \overset{0}{-} 110 \overset{1}{-} 101 \overset{1}{-} 011 \overset{0}{-} 110 \overset{0}{-} 100 \overset{0}{-} 000$

(DRUM ALSO NOT UNIQUE)