UNIVERSITY OF MANITOBA

DATE: February 4, 2015
COURSE: MATH 2400
EXAMINATION: Graph Theory

MIDTERM I
TITLE PAGE
TIME: 60 minutes
EXAMINER: M. Davidson

FAMILY NAME: (Print in ink) ______________________

GIVEN NAME(S): (Print in ink) ______________________

STUDENT NUMBER: ____________

SIGNATURE: (in ink) ______________________
(I understand that cheating is a serious offense)

SOLUTIONS

INSTRUCTIONS TO STUDENTS:

This is a 60 minute exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 7 pages of questions. Please check that you have all the pages.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 80 points.

Answer questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	20	
2	12	
3	10	
4	8	
5	8	
6	12	
7	10	
Total:	80	

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[20] 1. Fill in the following table according to these directions:

In the column labeled *vertices*, write the number of vertices of the graph.

In the column labeled *edges*, write the number of edges of the graph.

In the column labeled *regular/degree*, if the graph is regular, write **yes** and then the degree of each vertex; otherwise just write **no**.

In the column labeled *Eulerian*, write **yes** if the graph is Eulerian, otherwise write **no**.

In the column labeled *Hamiltonian*, write **yes** if the graph is Hamiltonian, otherwise write **no**.

	vertices	edges	regular/degree	Eulerian	Hamiltonian
K_7	7	21	yes - 6	yes	yes
C_{16}	16	16	yes - 2	yes	yes
P_{12}	12	11	no	no	no
$K_{6,10}$	16	60	no	yes	no
Q_4	16	32	yes - 4	yes	yes
K_8	8	28	yes - 7	no	yes
$K_{7,7}$	14	49	yes - 7	no	yes
N_{11}	11	0	yes - 0	no	no

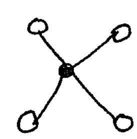
[12] 2. For each of the following, draw a graph on n vertices that satisfies the given properties if one exists. If no such graph exists, clearly show why not.

(a) $n = 6$; isomorphic to its complement.

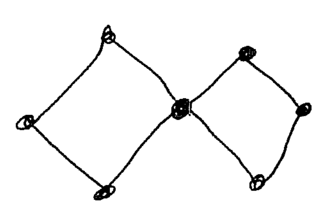
No such graph exist. K_6 has 15 edges. If a graph is isomorphic to its complement, then they must have the same number of edges. So the number of edges in the complete graph (on the same number of vertices) must be even.

(b) $n = 5$; complete bipartite with no vertices of degree 3.

$K_{1,4}$



(c) $n = 7$; Eulerian but not Hamiltonian



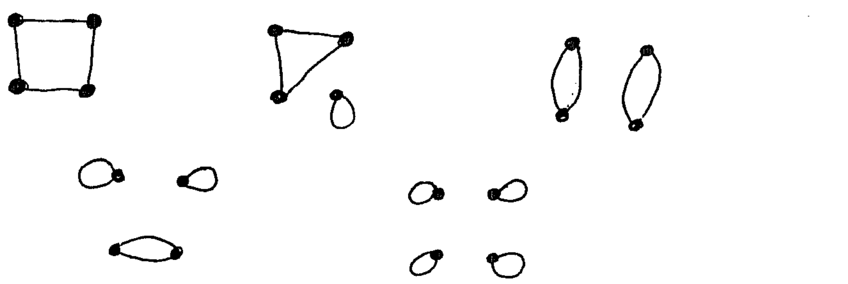
(There are other graphs that will satisfy these conditions)

(d) $n = 6$; simple with degree sequence $\{1, 2, 2, 3, 5, 5\}$

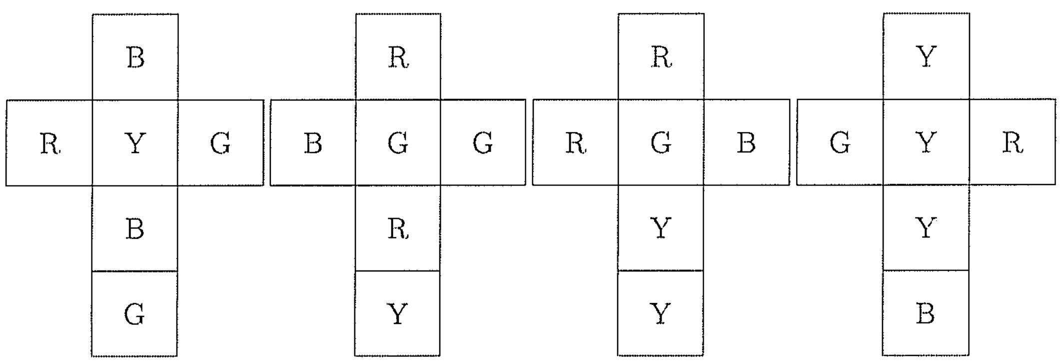
No such graph exists.
If G is simple on 6 vertices, and a vertex has degree 5, then it must be adjacent to all other vertices. If there are two vertices of degree 5, then all vertices must have degree 2 or greater. Hence there cannot be a vertex of degree 1.

[10] 3. (a) List all non-isomorphic 2-regular graphs on four vertices. (Draw the graphs.)

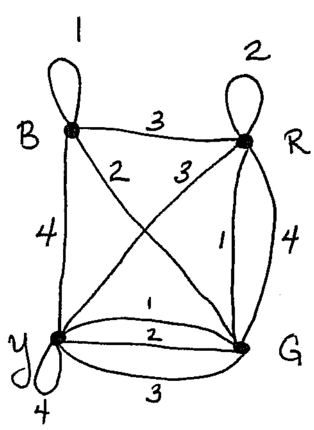
There are 5 2-regular graphs on 4 vertices.



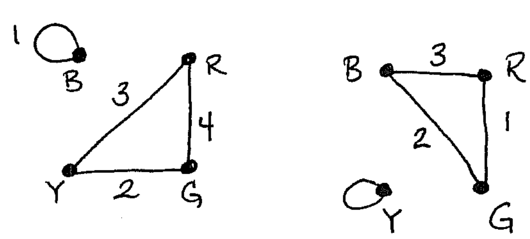
(b) Solve the following four cube problem (If a solution exists). Include the graph of the problem, as well as the graphs that give the solution to the problem. (If no solution exists, explain how you know this.)



cube 1 cube 2 cube 3 cube 4

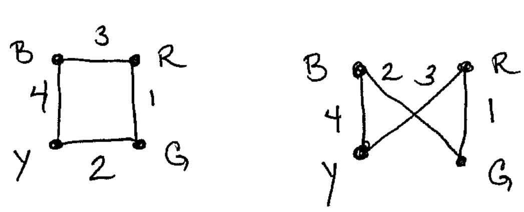


Solution:



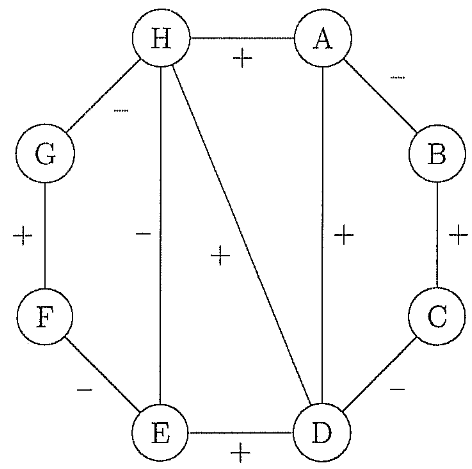
Note:

There were other graphs that were 2 regular, had one edge from each cube, but could not be paired for a solution.



[8] 4. For the following signed graphs, decide if they are balanced or not. (If it is balanced, include the sets that show it is balanced. If it is not balanced, give an explanation for how you know it is not.)

(a)



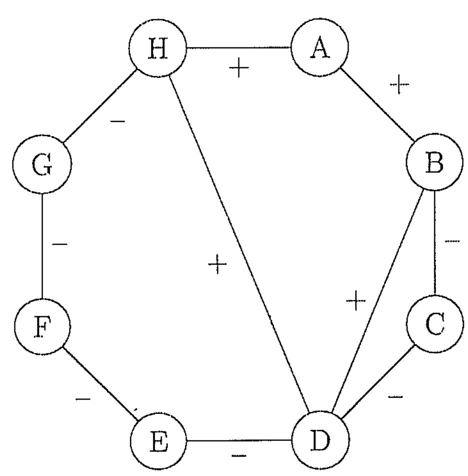
This signed graph is not balanced.

cycle H-D-E-H has one negative edge. (Odd)

Also Acceptable:

cycle H-E-F-G-H has 3 negative edges. (odd)

(b)



This signed graph is balanced.

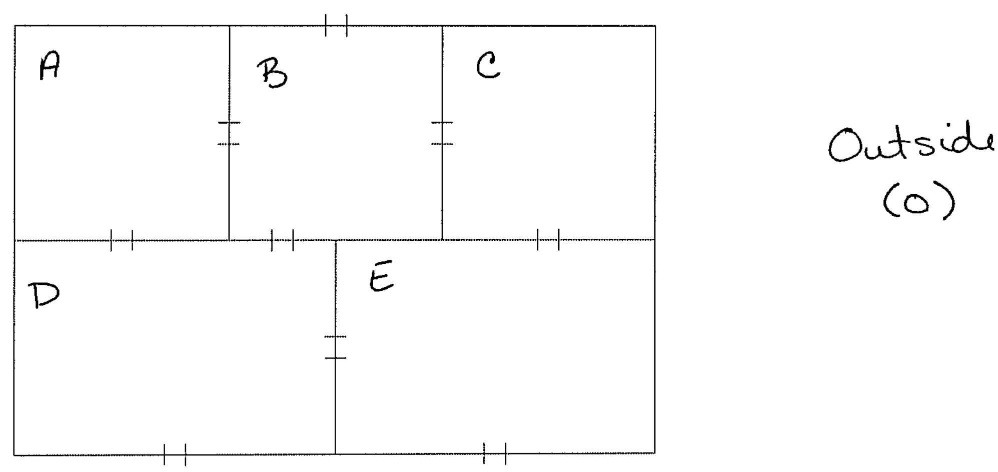
Here are the sets

{H, A, B, D, F}

and

{C, E, G}

[8] 5. The following diagram shows a house plan with doors indicated by small parallel lines. Someone wants to find a way to walk through the house (and outside - the big blue room) passing through every door exactly once. Can this be done? Describe how this problem can be solved with a graph. Mention what the vertices represent, what the edges represent, and what a solution is with respect to the graph. Build a graph to solve the problem and solve it.



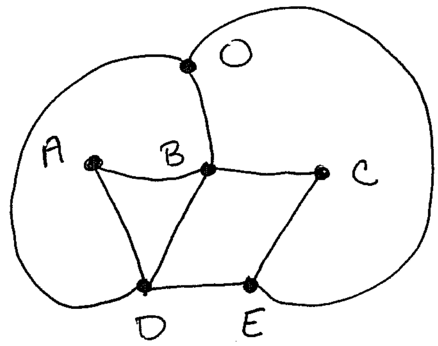
Let the rooms (plus outside) be the vertices of the graph.

Let the doors be the edges of the graph.

(Two room are adjacent in the graph if there is a door between the rooms)

A solution (going through all doors) would be an Eulerian trail or a Semi-Eulerian trail.

Graph

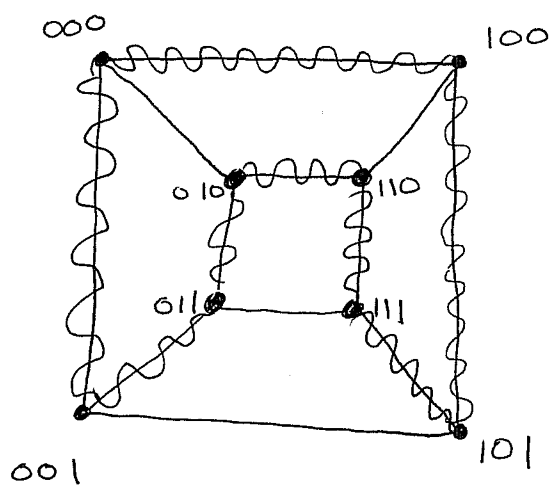


Solution

O-B-A-D-B-C-E-O-D-E

(There are other solutions)

[12] 6. (a) In the space below, draw the graph Q_3. Be sure to properly label the vertices.



(b) Find a Gray code on words of length 3.

000
001
011
010
110
111
101
100

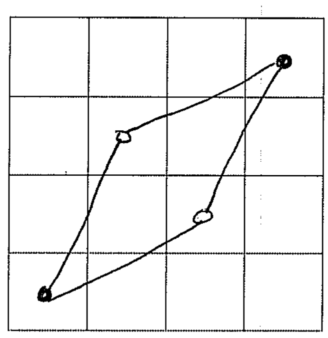
(There are other Gray codes)

(c) Indicate which edges are associated with the Gray code in the graph drawn above. (You may do this by drawing a squiggle over the appropriate lines.)

(d) What do those edges form (in terms of the graph)?

The edges from the Gray code form a Hamiltonian Cycle.

[10] 7. (a) Explain why there is no Knights tour on a 4×4 board.



The degree of each corner vertex is 2, so if a Hamiltonian cycle existed then those edges would be in it. This forms a closed loop that cannot be extended to include all squares (vertices).

(b) Explain why there is no solution to the knights tour problem on a 5×5 board.

The graph of the Knights tour problem is bipartite. A bipartite graph can only have a Hamiltonian cycle if the sets are the same size. A 5x5 board would have a graph with 25 vertices (odd) so the sets cannot be the same size; Hence no Hamiltonian cycle exists.

(c) In each square of the following 5×5 board, write the degree of the vertex associated with that square in the graph used in the Knights tour problem.

2	3	4	3	2
3	4	6	4	3
4	6	8	6	4
3	4	6	4	3
2	3	4	3	2

Note: The six (6) indicated squares will cover the entire board via symmetry