MATH 2030 – Combinatorics 1

Worksheet 10

1. Let $S_n = \{1, 2, \ldots, n\}$. Let X be the set of subsets of S_n, and we consider the relation $\mathbf{R}$ to be set containment ($A\mathbf{R}B$ if and only if $A \supset B$). Show that $(X, \mathbf{R})$ is a boolean algebra.

2. Show that if (X, R) is a boolean algebra and xRy, then $y \vee (x \wedge z) = x \wedge (y \vee z)$.

3. For each of the following, find (if possible) a recurrence relation for the described value:

   (a) (Tower of Hanoi - variant)
   Consider n circular disks (having different diameters) with holes in their centers. These disks are stacked on the first of three pegs with no disk resting upon one of a smaller diameter. A disk that is at the top of the stack on a peg can be moved to an *adjacent* peg provided it is not placed on top of a disk with a larger diameter. Let a_n be the minimum number of moves to transfer these disks to the last peg?

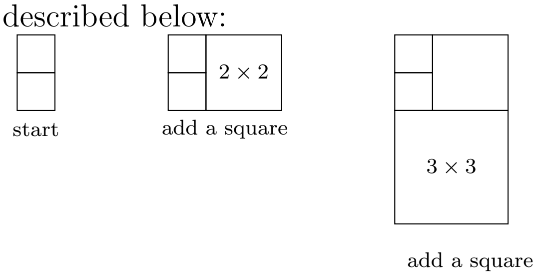
   (b) (Tower of Hanoi - variant)
   Consider n circular disks (having different diameters) with holes in their centers. These disks are stacked on the first of four pegs with no disk resting upon one of a smaller diameter. A disk that is at the top of the stack on a peg can be moved to another peg provided it is not placed on top of a disk with a larger diameter. Let a_n the minimum number of moves to transfer these disks to the last peg?

   (c) Let a_n be the number of binary words with no consecutive 0's.

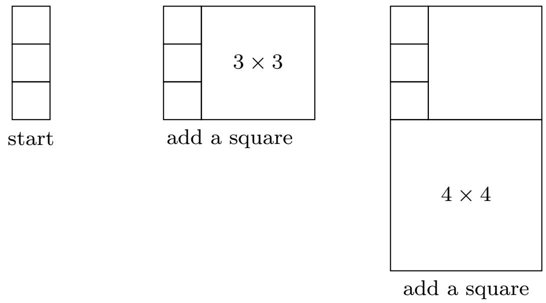
   (d) Let a_n be the number of ways to write $2n$ with n summands.

   (e) Suppose we have blobs that produce more blobs once in their lifetime. A Blue blob will produce a Green blob, and a Green blob will produce one Blue blob and one Green blob. We start a blob culture with one Blue blob and one Green blob. Let a_n be the number of blobs in the nth generation.

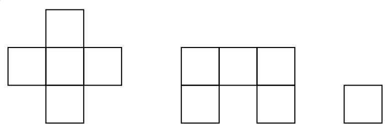
   (f) Let a_n be the side length of the nth square to be added to the sequence described below:

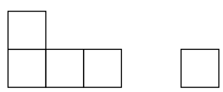


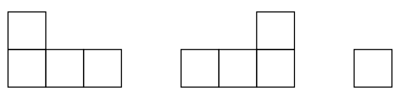
(g) Let a_n be the side length of the nth square to be added to the sequence described below:

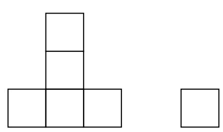


(h) Let a_n be the number of ways to tile a $3 \times n$ checkerboard with the following tiles.

i. 

ii. 

iii. 

iv. 

4. Solve the following recurrences:

   (a) $a_n = (-1)a_{n-1} + 6a_{n-2}$, where $a_0 = 3, a_1 = -14$.

   (b) $a_n = 6a_{n-1} - 5a_{n-2}$, where $a_0 = 1, a_1 = 17$.

   (c) $a_n = 4a_{n-1} - 4a_{n-2}$, where $a_0 = 3, a_1 = 16$.

   (d) $a_n = 4a_{n-1} + 21a_{n-2}$, where $a_0 = 7, a_1 = 19$.

   (e) $a_n = 9a_{n-1} - 26a_{n-2} + 24a_{n-3}$, where $a_0 = -2, a_1 = -1, a_2 = 11$.

   (f) $a_n = 5a_{n-1} - 3a_{n-2} - 9a_{n-3}$, where $a_0 = 7, a_1 = 2, a_2 = 29$.