Name: ______________________

Student ID: ______________

MATH 2030 – Combinatorics 1

Quiz 2

Answer all questions and show all your work. (Total Marks: 20)
You have 20 minutes to complete the quiz.

1. (8 points) An athlete is in training for 30 (consecutive) days. She will train at least once a day. At the end of the 30 days, her average number of training sessions will not be above 1.5 times a day. Show that there are consecutive days where she trained a total of 13 times.

Let a_i be the number of times that the athlete trained after day i. Since she trained at least once a day we get

$$1 \leq a_1 < a_2 < a_3 \cdots < a_{29} < a_{30}.$$

Also, since the average was not above 1.5 times a day $a_{30} \leq 45$.

So

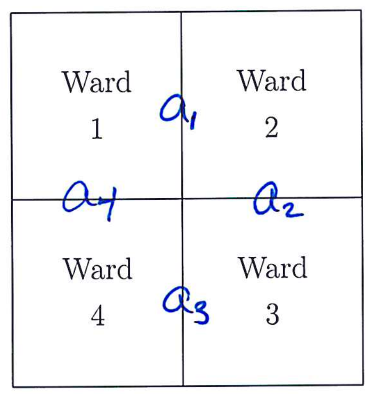
$$1 \leq a_1 < a_2 < a_3 \cdots < a_{29} < a_{30} \leq 45$$

also (add 13 to everything)

$$14 \leq a_1+13 < a_2+13 < a_3+13 < \cdots < a_{29}+13 < a_{30}+13 \leq 58$$

Now there are 60 numbers (a_i and a_j+13 ; 30 of each) and every number is between 1 and 58 (inclusive). So by the P.H.P there is a repeated number. Since all the a_i's are distinct (strictly increasing) and all the (a_j+13)'s are distinct (also strictly increasing) we must have some $a_i = a_j + 13$. Hence on days $j+1, j+2, \dots, i$ the athlete trained exactly 13 times.

2. (12 points) There is a town called Squaresville that will be having an election. They will elect a representative in each of their 4 wards. The representatives will come from one of the 4 political parties (Party A, Party B, Party C, and Party D). The wards suffer from rivalry, and would prefer to not have a representative from the same political party as either of the two neighbouring wards. How many ways can the people of Squaresville elect representatives so that neighbouring wards have different party affiliations?



Let a_1 be the condition that Ward 1 and Ward 2 have representatives of the same party.

a_2 . . . Ward 2 and Ward 3 . . .

a_3 . - - - - . Ward 3 and Ward 4

a_4 . . . . . . Ward 4 and Ward 1.

Now $N(\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4) = N - S_1 + S_2 - S_3 + S_4$

$N = 4^4$

$N(a_1) = 4^3$ (note: one choice for Ward 1 & Ward 2)

by symmetry $N(a_2) = N(a_3) = N(a_4) = N(a_1) = 4^3$; So $S_1 = 4 \cdot 4^3$

$N(a_1 a_2) = 4^2$ (Note: one choice for Ward 1 & 2 & 3)

by symmetry $N(a_2 a_3) = N(a_3 a_4) = N(a_4 a_1) = N(a_1 a_2) = 4^2$

also $N(a_1 a_3) = N(a_2 a_4) = 4^2$ ($N(a_1 a_3) = 4^2$; one choice for Ward 1 & 2, one for 3 & 4)

So $S_2 = 6 \cdot 4^2$

$N(a_1 a_2 a_3) = 4$ (all wards need same party for this)

By Symmetry $S_3 = 4 \cdot 4$

$N(a_1 a_2 a_3 a_4) = 4$.

So there are $4^4 - 4(4^3) + 6 \cdot 4^2 - 4 \cdot 4^2 + 4 = 5 \cdot 4^2 + 4 = 21(4) = 84$.