Name: Solutions

Student ID: ____________

MATH 2030 – Combinatorics 1

Quiz 3

Answer all questions and show all your work. (Total Marks: 20)
You have 20 minutes to complete the quiz.

1. (10 points)

   (a) Four couples are seated at a panel table (all in a row). How many ways can they be seated so that no couple sits together?

   (b) Four couples are seated at a panel table. How many ways can they be seated so that exactly one couple sits together?

We will refer to the couples as couple 1, couple 2, couple 3 and couple 4.

Let a_i be the property that couple i sits together.

(Note: couple i consists of two people, we'll call them A_i and B_i).

N is the number of permutations of 8 people, $N = 8!$

$N(a_1) = 2(7!)$ { we consider permutions with the "block" A_1B_1 and also with the "block" B_1A_1 }

by symmetry $S_1 = \binom{4}{1} \cdot 2 \cdot 7!$

$N(a_1a_2) = 2^2 \cdot 6!$ by symmetry $S_2 = \binom{4}{2} 2^2 \cdot 6!$

$N(a_1a_2a_3) = 2^3 \cdot 5!$ by symmetry $S_3 = \binom{4}{3} 2^3 \cdot 5!$

$N(a_1a_2a_3a_4) = 2^4 \cdot 4! = S_4$

a) $N(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4) = 8! - \binom{4}{1} \cdot 2 \cdot 7! + \binom{4}{2} \cdot 2^2 \cdot 6! - \binom{4}{3} 2^3 \cdot 5! + 2^4 \cdot 4!$

b) $e_1 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3 - \binom{4}{3} S_4$

$= \binom{4}{1} \cdot 2 \cdot 7! - 2 \cdot \binom{4}{2} 2^2 \cdot 6! + 3 \binom{4}{3} \cdot 2^3 \cdot 5! - 4(2^4 \cdot 4!)$.

2. (10 points)

How many permutations of 12345678 have exactly two of the following as (consecutive) subsequences?
12, 23, 34, 56, 78.

Let a_1 be the property that the permutation contains 12
a_2 23
a_3 34
a_4 56
a_5 78

$N(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4\bar{a}_5) = N - S_1 + S_2 - S_3 + S_4 - S_5$

$N = 8!$

These calculations were not necessary

$N(a_1) = 7!$, by symmetry $S_1 = \binom{5}{1} \cdot 7!$

$N(a_1a_2) = N(a_2a_3) = 6!$ {one LARGE piece, 123 & 234 resp with rest}

All other $N(a_ia_j) = 6!$ {two double pieces, and 4 singles)

$\therefore S_2 = \binom{5}{2} 6!$

$N(a_1a_2a_3) = 5!$ { Pieces 1234, 5, 6, 7, 8}

$N(a_1a_2a_i) = N(a_2a_3a_i) = 5!$ { Triple {123 or 234}, double, 3 singles}

other $N(a_ia_ja_k) = 5!$ { Three doubles, 2 singles}

$\therefore S_3 = \binom{5}{3} 5!$

$N(a_ia_ja_ka_l) = 4!$ { One Quadruple, one double, 2 singles (eg $a_1a_2a_3a_4$ permutes 1234, 56, 7, 8) OR one triple, 2 doubles, 1 single (eg $a_1a_2a_4a_5$ permutes 123, 56, 78, 4)

$S_4 = \binom{5}{4} 4!$

$N(a_1a_2a_3a_4a_5) = 3! = S_5$ { permute 1234, 56 and 78 }.

$$e_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 - \binom{5}{3} S_5$$
$$= \binom{5}{2} 6! - \binom{3}{1}\binom{5}{3} 5! + \binom{4}{2}\binom{5}{4} 4! - \binom{5}{3} 3!$$