MATH 2030 Combinatorics I Worksheet 10

1. We want to show this forms a complemented distributive lattice.

- For subsets A_1 and A_2 ; $A_1 \cup A_2$ is the smallest subset that contains both A_1 and A_2 ; hence $A_1 \vee A_2 = A_1 \cup A_2$; Similarly $A_1 \wedge A_2 = A_1 \cap A_2$. So this is a lattice with $S_n = \hat{1}$ and $\emptyset = \hat{0}$.

- $A_1 \vee (A_2 \wedge A_3) = A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap (A_1 \cup A_3)$
$= (A_1 \vee A_2) \wedge (A_1 \vee A_3)$

and

$A_1 \wedge (A_2 \vee A_3) = A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$
$= (A_1 \wedge A_2) \vee (A_1 \wedge A_3)$

This shows distributivity.

- For $A_1 \subseteq S_n$; Let $\overline{A_1} = S_n \setminus \{A_1\}$, so $\overline{A_1}$ is the set of elements of S_n that are not in A_1.
So $A_1 \cup \overline{A_1} = S_n \Rightarrow A_1 \vee \overline{A_1} = \hat{1}$
and $A_1 \cap \overline{A_1} = \emptyset \Rightarrow A_1 \wedge \overline{A_1} = \hat{0}$.

Hence it is complemented.

2.

$y \vee (x \wedge z)$ (dist.)
$= (y \vee x) \wedge (y \vee z)$ since xRy $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \to y \vee x = x$
$= x \wedge (y \vee z)$

* note we did not use complemented in this proof.

3(a) We use this method :

move the 1 to $n-1$ stack to the last peg (a_{n-1})

move the n disk to the second peg (1)

move the 1 to $n-1$ stack to first peg (a_{n-1})

move the n disk to the last peg (1)

move the 1 to $n-1$ stack to the last peg (a_{n-1})

So $a_n = 3a_{n-1} + 2$

$a_1 = 2$

< I believe this is optimal. >

(b) We use this method

move the 1 to $n-2$ stack to the second peg (a_{n-2})

move the $n-1$ to the third peg (1)

move the n to the last peg (1)

move the 1 to $n-2$ stack to the last peg (a_{n-2})

So $a_n = 2a_{n-2} + 2$

$a_1 = 1$

$a_2 = 3$

< I do not know if this is optimal. >

(c) If there is an $n-1$ length word that has no consecutive 0's, then adding a 1 to the end leave a word with no consecutive 0's. (a_{n-1})
If we attempted to add a 0 to the end of a $n-1$ length word, then it would only have no consecutive 0's if it ended in a 1. So this is the same as appending 10 to an $n-2$ length word (a_{n-2})

$$a_n = a_{n-1} + a_{n-2} \qquad a_1 = 2$$
$$a_2 = 3$$

(d) {Partial answer}
We consider counts up to the order of summands, ie $1+3 = 3+1$ for this.

If we have a summand of $2(n-1)$ in $n-1$ summand, we can suppose the summands are in non-decreasing order.
From this we can create $2n$ by adding a new summand $(1+)$ to the front, and up the last one by 1.
{There are a few of these that are missed, like $2+2+\cdots+2$ }

$$a_n = a_{n-1} + 1 + k$$

But I don't know what k depends upon.

e.

```
    G   B            2 blob
   / \  |
  G  B  G            3 blobs
 / \ |  / \
G  B G  G  B         5 blobs
```

In any generation, the number of green blobs is equal to the number of blobs in the last generation (all blobs produce a green). The number of blue blobs in a generation is the number of green blobs in the previous generation, which is exactly the total number of blobs in the generation before the previous one.

$$a_n = a_{n-1} + a_{n-2} \qquad a_1 = 2, \quad a_2 = 3$$

f. The square added has length of the longest side of the previous rectangle, which will be the sum of the last two squares added.

$$a_n = a_{n-1} + a_{n-2} \qquad a_1 = 2, \quad a_2 = 3$$

{at this point, notice that c, e and f all have the same answer.}

g) this is the same formula as (f) with different boundary conditions.

$$a_n = a_{n-1} + a_{n-2} \qquad a_1 = 3, \quad a_2 = 4$$

h) ⟨See sheet with forms⟩

(i) $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3} + 4a_{n-4} + a_{n-5}$

$a_1 = 1 \quad a_2 = 3 \quad a_3 = 8 \quad a_4 = 21 \quad a_5 = 51$

(ii) and (iii) have no recurrance relation, as we can find a way to tile a strip of arbitrary/variable length.

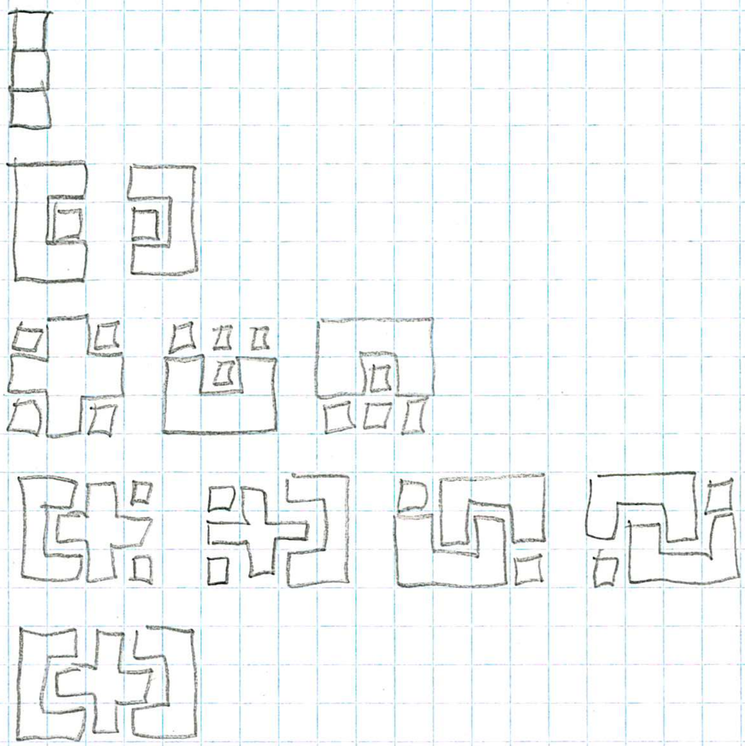
These sets of tiles will nicely tile a $2 \times n$ Checkerboard without running into the same difficulty.

(iv) $a_n = a_{n-1} + 4a_{n-3}$

$a_1 = 1 \quad a_2 = 1 \quad a_3 = 5$

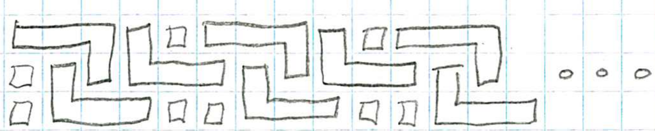
(FORMS)

h(i)

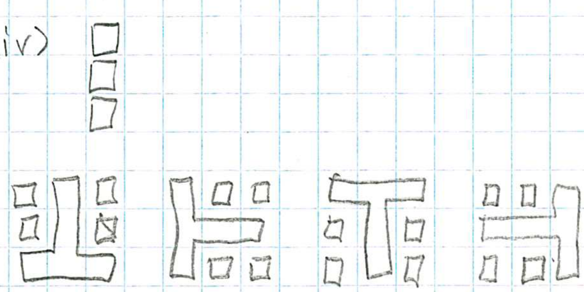


h(ii) and (iii)

cannot be done



h(iv)



4 (a) $x^2 = -x + 6$

$x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$

$x = -3 \quad x = 2$

$a_n = C_1 2^n + C_2 (-3)^n$

$a_0 = C_1 + C_2 = 3$ $\Rightarrow$ $C_1 = {-25}/{-5} = 5$ $\left\{ \text{Cramer's Rule} \right\}$

$a_1 = 2C_1 - 3C_2 = -14$ $\quad C_2 = {-20}/{-5} = 4$

So $a_n = 5 \cdot 2^n + 4 \cdot (-3)^n$

(b) $x^2 = 6x - 5$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

$x = 5 \quad x = 1$

$a_n = C_1 \cdot 1^n + C_2 \cdot 5^n$

$a_0 = C_1 + C_2 = 1$ $\Rightarrow$ $C_1 = -3$

$a_1 = C_1 + 5C_2 = 17$ $\quad C_2 = 4$

So $a_n = -3 \cdot 1^n + 4 \cdot 5^n = 4 \cdot 5^n - 3$

(c) $a_n = 4a_{n-1} - 4a_{n-2}$

$x^2 = 4x - 4$

$x^2 - 4x + 4 = 0$

$(x-2)(x-2) = 0$

$x = 2, 2$

$a_n = C_1 \cdot 2^n + C_2 \cdot n 2^n$

$a_0 = C_1 = 3$

$a_1 = 2C_1 + 2C_2 = 16$

$\Rightarrow$ $C_1 = 3$, $C_2 = 5$

$a_n = 3 \cdot 2^n + 5n \cdot 2^n$

d) $a_n = 4a_{n-1} + 21a_{n-2}$

$x^2 = 4x + 21$

$x^2 - 4x - 21 = 0$

$(x-7)(x+3) = 0$

$x = 7 \quad x = -3$

$a_n = C_1 7^n + C_2 (-3)^n$

$a_0 = C_1 + C_2 = 7$

$a_1 = 7C_1 - 3C_2 = 19$

$\Rightarrow$ $C_1 = 4$, $C_2 = 3$

$a_n = 4 \cdot 7^n + 3(-3)^n$

e) $a_n = 9a_{n-1} - 26a_{n-2} + 24a_{n-3}$

$x^3 = 9x^2 - 26x + 24$

$x^3 - 9x^2 + 26x - 24 = 0$

$(x-2)(x-3)(x-4) = 0$

$x = 2 \quad x = 3 \quad x = 4$

$a_n = C_1 2^n + C_2 3^n + C_3 4^n$

$$\begin{aligned} a_0 &= C_1 + C_2 + C_3 = -2 \\ a_1 &= 2C_1 + 3C_2 + 4C_3 = -1 \\ a_3 &= 4C_1 + 9C_2 + 16C_3 = 11 \end{aligned} \quad \Rightarrow \quad \begin{aligned} C_1 &= -3 \\ C_2 &= -1 \\ C_3 &= 2 \end{aligned}$$

$a_n = -3 \cdot 2^n + (-1) \cdot 3^n + 2 \cdot 4^n$

f) $a_n = 5a_{n-1} - 3a_{n-2} - 9a_{n-3}$

$x^3 = 5x^2 - 3x - 9$

$x^3 - 5x^2 + 3x + 9 = 0$

$(x+1)(x-3)(x-3) = 0$

$x = -1 \quad x = 3, 3$

$a_n = C_1(-1)^n + C_2 \cdot 3^n + C_3 n \cdot 3^n$

$$\begin{aligned} a_0 &= C_1 + C_2 = 7 \\ a_1 &= -C_1 + 3C_2 + 3C_3 = 2 \\ a_2 &= C_1 + 9C_2 + 18C_3 = 29 \end{aligned} \quad \Rightarrow \quad \begin{aligned} C_1 &= 5 \\ C_2 &= 2 \\ C_3 &= 1/3 \end{aligned}$$

$a_n = 5 \cdot (-1)^n + 2 \cdot 3^n + \frac{1}{3} n 3^n$