MATH 2030 – Combinatorics 1

Worksheet 1

Answer all questions and show all your work. (Total Marks: 30)
You have 20 minutes to complete the quiz.

1. (20 points) In the following, find the count. Answers should be in exponential or factorial notation.
   (a) How many different ways can we arrange the letters in QUESTION?
   (b) We want to make 3 letter words out of the letters in QUESTION. Assuming any arrangement of 3 letters is considered a word, how many distinct words can we make.
   (c) How many ternary words are there that have length 8?
   (d) Lisa goes to the snack bar, and finds out there are 8 kinds of chips and 3 kinds of chocolate bars. How many ways can she purchase a snack for herself, and one for her son?
   (e) Jim has been dealt 8 cards and must discard 3. How many ways can he choose his discards?

2. For each of the following, identify if and how the counts are different. If possible, express the count(s).
   (a) i. Shelley has gone to an elementary school to do a presentation for 3 different classrooms. She has brought with her 8 different games related to the topic she will be discussing, and she wants to leave one game in each of the classrooms she visits. In how many ways can she do this?
   ii. Shelley has gone to an elementary school to do a presentation for 3 different classrooms. She has brought with her 8 different games related to the topic she will be discussing, and she wants to leave 3 of the games behind in the school. In how many ways can she do this?
   (b) i. How many ways can grades be assigned to a class with 8 students if the only grades being assigned are $A^+, A,$ or B^+?
   ii. Mike's transcript for the 8 courses he took last year contained the grades $A^+, A,$ and B^+, (and only those grades). How many different transcripts are possible for Mike's 8 courses?

(c) i. Lisa goes to the snack bar, and finds out there are 8 snacks options. She wants to buy a snack for herself and her two friends. Both of her friends are weird about snack, they do not like to eat the same kind of snack as anyone else in their group. How many ways can Lisa buy appropriate snacks?

ii. Lisa goes to the snack bar, and finds out there are 8 snack options. She wants to buy a snack for herself and her two friends. Both of her friends are weird about snack, they do not like to eat the same kind of snack as anyone else in their group. She also has her son with her, and he needs a snack that is the same as someone in the group. How many ways can Lisa buy appropriate snacks?

3. For each of the following

   (a) Write down the binomial identity being described.

   (b) Give a combinatorial proof of the identity.

   (c) Give an algebraic proof of the identity.

   i. In a high school class of n students, a bunch of give-aways are brought in to a special assembly. No student will get more than one of the giveaways. There are k_1 red shirts with the schools motto, and k_2 blue hats with the school crest. The principle wants to give away the red shirts first, and then give away the blue hats; but the Drama teacher thinks they should first give away the blue hats, and then the red shirts.

   How many different way can they give out the give-aways?

   ii. Wendy has made a series of n bowl in her pottery class, which she has labelled $1, \ldots, n$. She has just bought a new red glaze, and she has enough to glaze m bowls. On k of those bowls, she would like to also add a glass bead on the inside. She is not sure if she should start by first picking all of the bowls that will get the red glaze, and then picking of them the k in which to put a glass bead; or if she should first pick the k that get the additional glass bead, and then choose the remaining bowls to be glazed red.

   How many different ways can she pick bowls that will be glazed red, some of which will also have a glazed glass bead?

1 (a) $8!$

(b) 8^3

(c) 3^8

(d) $(8+3)^2 = 11^2$

(e) $\binom{8}{3} = \dfrac{8!}{5!\,3!}$

2 (a) These are distinct. In case (i), the order of the games matter, and in case (ii) it will not matter.

(i) $8 \cdot 7 \cdot 6 = \dfrac{8!}{5!}$ (ii) $\binom{8}{3} = \dfrac{8!}{5!\,3!}$

(b) These are distinct. In case (i), we allow the possibility that the grades are drawn from a subset of $\{A^+, A, B^+\}$. For example, all the grades could be A^+. In case (ii), there needs to be at least one of each grade.

(i) 3^8 (ii) This will be covered when we do inclusion / exclusion.

(c) These are the same. In case (ii), the only possible snack that can be repeated is Lisa's, leaving only one way to choose her son's snack.

(i) $\binom{8}{3} = \dfrac{8!}{5!\,3!}$ (ii) $\binom{8}{3} \cdot 1 = \dfrac{8!}{5!\,3!}$

3: (a) $\binom{n}{k_1}\binom{n-k_1}{k_2} = \binom{n}{k_2}\binom{n-k_2}{k_1}$

(b) Given an n-set, we wish to choose two distinct, disjoint subsets; one of size k_1 and one of size k_2.

We can count the number of ways this can be done, using two different methods.

Firstly, we pick the subset of size k_1. Then, out of the remaining $n-k_1$ elements we choose the subset of size k_2. There are $\binom{n}{k_1}\binom{n-k_1}{k_2}$ ways we can do this.

Secondly, we pick the subset of size k_2 first, then the subset of size k_1 out of the remaining $n-k_2$ elements. There are $\binom{n}{k_2}\binom{n-k_2}{k_1}$ ways we can do this.

Hence $\binom{n}{k_1}\binom{n-k_1}{k_2} = \binom{n}{k_2}\binom{n-k_2}{k_1}$,

as they both represent the number of ways to choose two disjoint subsets of size k_1 and k_2 from an n-set.

3i

(c) $$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!\cancel{(n-k_1)!}} \cdot \frac{\cancel{(n-k_1)!}}{(n-k_1-k_2)!\,k_2!}$$

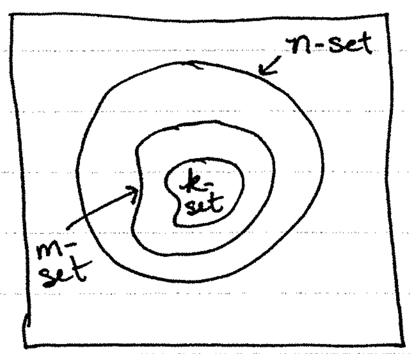
$$= \frac{n!}{k_1!\,(n-k_1-k_2)!\,k_2!} = \frac{n!\,(n-k_2)!}{(n-k_1-k_2)!\,(n-k_2)!\,k_1!\,k_2!}$$

$$= \frac{n!}{(n-k_2)!\,k_2!} \cdot \frac{(n-k_2)!}{(n-k_2-k_1)!\,k_1!} = \binom{n}{k_2}\binom{n-k_2}{k_1}$$

3ii (a) $$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

(b) Given an n-set, we wish to choose a subset of size m, and from that m-set, we wish to choose a subset of size k.

DIAGRAM



We can count the number of ways to do this in two ways.

The first way, we start by choosing the m-set as a subset of the n-set. We then choose the k-set as a subset of the m-set. There are $\binom{n}{m}\binom{m}{k}$ ways to do this

3i (b) cont

The second way, we start by choosing from n the k elements that form the k-set that is a subset of the m-set, then we choose the remaining elements of the m-set. There are $\binom{n}{k}\binom{n-k}{m-k}$ ways to do this.

Hence $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, as they are

both the number of ways to choose an m-element subset of an n-set, and a k-element subset of the m-set.

(c) $$\binom{n}{m}\binom{m}{k} = \frac{n!}{m!(n-m)!} \cdot \frac{m!}{k!(m-k)!} = \frac{n!}{(n-m)!(m-k)!k!}$$

$$= \frac{n!\,(n-k)!}{(n-m)!(m-k)!k!(n-k)!} = \frac{n!}{(n-k)!k!} \cdot \frac{(n-k)!}{(n-m)!(m-k)!}$$

$$= \frac{n!}{(n-k)!k!} \cdot \frac{(n-k)!}{((n-k)-(m-k))!(m-k)!} = \binom{n}{k}\binom{n-k}{m-k}$$