MATH 2030 Worksheet 4 Solutions

1(a) $\sum_{i=1}^{25} \binom{25}{i} 3^i = \sum_{i=1}^{25} \binom{25}{i} 3^i 1^{25-i} = (3+1)^{25} = 4^{25} = 2^{50}$

hence $x = 2$.

(b) $\sum_{i=1}^{10} \binom{10}{i} 7^i = \sum_{i=1}^{10} \binom{10}{i} 7^i 1^{10-i} = (7+1)^{10} = 8^{10} = 2^{30} = 4^{15}$

hence $x = 4$.

(c) $\sum_{i=1}^{30} \binom{30}{i} 5^i 2^{30-i} = (5+2)^{30} = 7^{30} = (7^2)^{15} = 49^{15}$

hence $x = 49$.

2. Let t_i be the number of elements in the largest subsequence that starts at the i^{th} element and is increasing. If there is some j for which $t_j \geq m+1$ then we know an increasing subsequence of length $m+1$ exists.

Otherwise, every t_i satisfies $1 \leq t_i \leq m$. Since there are $mn+1$ such t_i's, we know that there is some value that is repeated at least $n+1$ times $\left\{ \text{P.H.P } \left\lfloor \frac{(mn+1)-1}{m} \right\rfloor = n \right\}$

Now let a_y and a_z $(y < z)$ be entries in the sequence where $t_y = t_z$. If $a_y < a_z$, then we could form a subsequence starting at a_y and attaching an increasing subsequence starting at a_z to form an increasing subsequence. This means that if $a_y < a_z$ then $t_y \geq t_z + 1$. So the $n+1$ elements of the sequence that all have the same t_i value form a decreasing subsequence.

3. We denote the largest value in the set A as a_m. We consider the values all of which are between 1 and n; The a_i where $a_i \in A$ and the $a_m - a_j$ where $a_j \in A \setminus \{a_m\}$. The first type are all distinct and there are at least $\lfloor \frac{n+2}{2} \rfloor$ of them. The second kind are also all distinct, and there are at least $\lfloor \frac{n+2}{2} \rfloor - 1$ of them. In total there are at least $n+1$ number from 1 to n, hence two must be the same so we must have $a_i = a_m - a_j$, or $a_i + a_j = a_m$ where a_i, a_j and $a_m \in A$ as desired.

4. Consider the sets

$$\{1\}, \{4, 100\}, \{7, 97\}, \ldots, \{49, 55\}, \{52\}$$

There are 18 such sets, and so if we pick 19 numbers, some set contains 2 numbers (we note that $\{1\}$ and $\{52\}$ each cannot contain two numbers). The sum of those two numbers is 104.

5.(a) We form the sets

$$\{1\}, \{3, 99\}, \{5, 97\}, \ldots, \{49, 53\}, \{51\}$$

There are 26 such sets (see above).

(b) The sets we want to consider are

$$\{2\}, \{4, 98\}, \{6, 96\}, \ldots, \{50, 52\},$$

We note that there are only 25 sets, so if we pick 26, then we guaruntee that some set contains two.