Worksheet 5
Soln sketches

1(a) a_1 is from Otherland
a_2 is Troll
a_3 is has purple hair

$N = 120$ $\quad S_1 = 55 + 42 + 48 = 145$
$S_2 = 20 + 17 + 13 = 50$
$S_3 = 5$

i) $N(\bar{a}_1 \bar{a}_2 \bar{a}_3) = 120 - 145 + 50 - 5 = 20$
There are 20 not from otherland, not trolls, not purple hair.

ii) $N(\bar{a}_1 \bar{a}_2 a_3) = N(a_3) - N(a_1 a_3) - N(a_2 a_3) + N(a_1 a_2 a_3)$

$= 48 - 17 - 13 + 5$
$= 23$

There are 23 with Purple Hair but not troll & not from Otherland.

1(b) $N = 65$

$S_1 = 21 + 35 + 28 + 32 = 116$

$S_2 = 13 + 10 + 9 + 12 + 17 + 14 = 75$

$S_3 = 4 + 6 + 5 + 7 = 22$

$S_4 = 0$

i) $N(\bar{a_i}) = 65 - 116 + 75 - 22 = 2$

There a 2 on clean up.

ii) $21 - (13 + 10 + 9) + (4 + 6 + 5)$
$= 4$

There are 4 who bring only hot dogs

iii) $35 - (13 + 12 + 17) + (4 + 6 + 7)$
$= 10$

only Fried Chicken

$28 - (10 + 12 + 14) + (4 + 5 + 7)$
$= 8$

only salad

$32 - (9 + 17 + 14) + (6 + 5 + 7)$
$= 10$

only dessert.

$\therefore$ Number of people who only bring one item are
$4 + 10 + 8 + 10 = 32$

{We will leave an alternate way for this}

2. Let a_i be the arrangement contains the subsequence 123

a_2 456

a_3 789

$N(\bar{a}_1 \bar{a}_2 \bar{a}_3) = N - S_1 + S_2 - S_3$

$N = 9!$

$N(a_1) = 7!$ (consider 123 as a whole number)

$N(a_2) = N(a_3) = 7!$ so $S_1 = 3 \cdot 7!$

$N(a_1 a_2) = 5!$ $S_2 = 3 \cdot 5!$

$N(a_1 a_3) = N(a_2 a_3) = 5!$

$N(a_1 a_2 a_3) = 3!$ $S_3 = 3!$

So $N(\bar{a}_1 \bar{a}_2 \bar{a}_3) = 9! - 3 \cdot 7! + 3 \cdot 5! - 3!$

3. Let a_1 be the arrangement contains 123.
a_2 . . . 345.
a_3 . . . 369.

$N(\bar{a}_1\ \bar{a}_2\ \bar{a}_3) = N - S_1 + S_2 - S_3$

$N = 9!$

$N(a_1) = N(a_2) = N(a_3) = 7!$ $S_1 = 3 \cdot 7!$

$N(a_1 a_2) = 5!$ (12345 as one NUMBER, with 6, 7, 8, 9)

$N(a_1\ a_3) = 5!$ (12369 . . . . 4, 5, 7, 8)

$N(a_2 a_3) = 0$

$N(a_1 a_2 a_3) = 0$

So $N(\bar{a}_1 \bar{a}_2 \bar{a}_3) = 9! - 3 \cdot 7! + 2 \cdot 5!$

4. Let a_1 be there were 6 or more Kit Kats
a_2 b . . . . . . Areos
a_3 . — — — — — — Smarties

$$N(\bar{a}_1\bar{a}_2\bar{a}_3) = N - S_1 + S_2 - S_3$$

$$N = \binom{10+3-1}{3-1} = \binom{12}{2}$$

$$N(a_1) = \binom{4+3-1}{3-1} = \binom{6}{2}$$ {assume 6 kitkats, choose the remaining 4 candies}

$$N(a_2) = N(a_3) = \binom{6}{2} \quad \therefore \quad S_1 = 3\cdot\binom{6}{2}$$

$N(a_1a_2) = 0$ (There cannot be more than 10)
$N(a_1a_3) = N(a_2a_3) = N(a_1a_2a_3) = 0.$

So $N(\bar{a}_1\bar{a}_2\bar{a}_3) = \binom{12}{2} - 3\binom{6}{2}.$

5. Let a_i be the number i does not appear.

($\therefore$ we have conditions $a_1, a_2, a_3, a_4, a_5, a_6$).

$$N(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4\bar{a}_5\bar{a}_6) = N - S_1 + S_2 - S_3 + S_4 - S_5$$

$N = 6^8$

$N(a_1) = 5^8 \quad (N(a_i) = 5^8) \therefore \quad S_1 = 6 \cdot 5^8$

$N(a_1 a_2) = 4^8 \quad (N(a_i a_j) = 4^8) \therefore \quad S_2 = \binom{6}{2} 4^8$

$N(a_1 a_2 a_3) = 3^8 \qquad S_3 = \binom{6}{3} \cdot 3^8$

$N(a_1 a_2 a_3 a_4) = 2^8 \qquad S_4 = \binom{6}{4} \cdot 2^8$

$N(a_1 a_2 \dots a_5) = 1^8 \qquad S_5 = \binom{6}{5} \cdot 1^8$

$S_6 = 0$

$$\therefore \; N(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4\bar{a}_5\bar{a}_6) = 6^8 - \binom{6}{1}5^8 + \binom{6}{2}4^8 - \binom{6}{3}3^8 + \binom{6}{4}2^8 - \binom{6}{5}1^8$$

$$\text{Hence probability} = \frac{6^8 - \binom{6}{1}5^8 + \binom{6}{2}4^8 - \binom{6}{3}3^8 + \binom{6}{4}2^8 - \binom{6}{5}1^8}{6^8}.$$

6. Let a_1 be the number is divisible by 2.

a_2 3

a_3 5

a_4 7

$$N(\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4) = N - S_1 + S_2 - S_3 + S_4$$

$N = 200$ $\quad N(a_1) = \left\lfloor \frac{200}{2} \right\rfloor = 100 \quad N(a_2) = \left\lfloor \frac{200}{3} \right\rfloor = 66$

$N(a_3) = \left\lfloor \frac{200}{5} \right\rfloor = 40 \quad N(a_4) = \left\lfloor \frac{200}{7} \right\rfloor = 28$

$S_1 = 234$

$N(a_1a_2) = \left\lfloor \frac{200}{6} \right\rfloor = 33 \quad N(a_1a_3) = \left\lfloor \frac{200}{10} \right\rfloor = 20 \quad N(a_1a_4) = \left\lfloor \frac{200}{14} \right\rfloor = 14$

$N(a_2a_3) = \left\lfloor \frac{200}{15} \right\rfloor = 13 \quad N(a_2a_4) = \left\lfloor \frac{200}{21} \right\rfloor = 9 \quad N(a_3a_4) = \left\lfloor \frac{200}{35} \right\rfloor = 5$

$S_2 = 94$

$N(a_1a_2a_3) = \left\lfloor \frac{200}{30} \right\rfloor = 6 \quad N(a_1a_2a_4) = \left\lfloor \frac{200}{42} \right\rfloor = 4$

$S_3 = 13$ $\quad N(a_1a_3a_4) = \left\lfloor \frac{200}{70} \right\rfloor = 2 \quad N(a_2a_3a_4) = \left\lfloor \frac{200}{105} \right\rfloor = 1$

$S_4 = 0$ $\quad N(a_1a_2a_3a_4) = \left\lfloor \frac{200}{210} \right\rfloor = 0$

$\therefore N(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4) = 200 - 234 + 94 - 13 = 47$

There are 47 such numbers.

7. Let a_1 be the number is divisible by 2 (See calcs from Q6)

a_2 ... 3

a_3 ... 5

a_4 ... 7

a_5 ... 11

a_6 ... 13

$$N(\bar{a}_i) = N - S_1 + S_2 - S_3 + S_4 - S_5 + S_6$$

$N = 200$

$N(a_5) = \left\lfloor \frac{200}{11} \right\rfloor = 18 \quad N(a_6) = \left\lfloor \frac{200}{13} \right\rfloor = 15$

$S_1 = 267$

$N(a_1a_5) = \left\lfloor \frac{200}{22} \right\rfloor = 9 \quad N(a_2a_5) = \left\lfloor \frac{200}{33} \right\rfloor = 6$

$N(a_3a_5) = \left\lfloor \frac{200}{55} \right\rfloor = 3 \quad N(a_4a_5) = \left\lfloor \frac{200}{77} \right\rfloor = 2$

$N(a_1a_6) = \left\lfloor \frac{200}{26} \right\rfloor = 7 \quad N(a_2a_6) = \left\lfloor \frac{200}{39} \right\rfloor = 5$

$N(a_3a_6) = \left\lfloor \frac{200}{65} \right\rfloor = 3 \quad N(a_4a_6) = \left\lfloor \frac{200}{91} \right\rfloor = 2 \quad N(a_5a_6) = 1$

$S_2 = 131$

$N(a_1a_2a_5) = \left\lfloor \frac{200}{66} \right\rfloor = 3 \; ; \left\lfloor \frac{200}{110} \right\rfloor = 1 \; ; \left\lfloor \frac{200}{154} \right\rfloor = 1$

$\left\lfloor \frac{200}{165} \right\rfloor = 1 \; ; \left\lfloor \frac{200}{231} \right\rfloor = 0 \; ; \left\lfloor \frac{200}{385} \right\rfloor = 0$

$\left\lfloor \frac{200}{78} \right\rfloor = 2 \quad \left\lfloor \frac{200}{130} \right\rfloor = 1 \quad \left\lfloor \frac{200}{182} \right\rfloor = 1$

$S_3 = 24$ $\quad \left\lfloor \frac{200}{195} \right\rfloor = 1 \quad \left\lfloor \frac{200}{273} \right\rfloor = 0 \quad \left\lfloor \frac{200}{455} \right\rfloor = 0$

$N(a_i a_5 a_6) = 0$

$S_4 = S_5 = S_6 = 0$ $\{$as $2 \times 3 \times 5 \times 7 > 200\}$

So $N(\bar{a}_i) = 200 - 267 + 132 - 24 = 41$

Hence for primes we add back in the 6 above, then subtract 1.

$\therefore$ 46 primes.

8. If there are d_4 derangements of 4 elements, then we would get $d_4 \times d_4$ (1234 would need to be deranged, and 5678 would need to be deranged).

Now $d_4 = 4! - \binom{4}{1}3! + \binom{4}{2}2! - \binom{4}{3}1! + \binom{4}{4}0!$
$= 24 - (4)(6) + (6)(2) - 4(1) + 1(1)$
$= 24 - 24 + 12 - 4 + 1$
$= 9$

Hence there are 81 derangements of 12345678 that have 1234 in the first 4 places.

9. If 5678 are in the first 4 places and 1234 are in the last 4 places, then it is a derangement. So there are $4! \times 4!$ such derangements.
So 576.