MATH 2030 WORKSHEET 7 SOLUTIONS

Q1 The objective of the challenge is to have disjoint subsets of the 7 numbers which have the same total.

Since there are 7 disks, there are $2^7 - 1 = 127$ distinct non-empty subsets.

Since the largest possible value on a disk is 20, the largest possible total would be $20 + 19 + 18 + 17 + 16 + 15 + 14 = 119$.

There are more subsets having some total (127) than there are possible totals (119 (from 1 to 119), so by the pigeon hole principle, there are two subsets (distinct) having the same total, say A and B.

To get disjoint subsets, we would remove from A and B all values that they had in common, leaving two subsets having the same total (and since A and B were disjoint, we know that these new subsets are not empty.)

Q2 Let b_i be the number of lattes consumed by Laura since the start of day 1 till the end of day i. Since she consumes at least one a day, if $i<j$ then $b_i < b_j$.

So we have

$$1 \leq b_1 < b_2 < b_3 < \cdots < b_{49} < b_{50} = 70$$

and so

$$25 \leq b_1+24 < b_2+24 < b_3+24 < \cdots < b_{49}+24 < b_{50}+24 = 94$$

There are 100 (50+50) numbers in the above lists, each number is between 1 and 94, so by the pigeonhole principle there is a number that is repeated.

Because of the strict inequality amongst the b_i's, we know that $b_{i_1} \neq b_{i_2}$ and also $b_{j_1}+24 \neq b_{j_2}+24$. So to get a number repeated we must have something in the first list equal to something in the second list. So there are i & j such that

$$b_i = b_j + 24$$
$$b_i - b_j = 24$$

This means that from the end of day i till the end of day j, 24 lattes were consumed. Hence on the consecutive days day $i+1$, day $i+2$, ..., day j there were exactly 24 lattes consumed.

Q3 Let a_1 be the property that the roll contains no 1's.

$\vdots$

Let a_8 be the property that the roll contains no 8's.

(alternately: for i from 1 to 8 let a_i be the property that the roll contains no i's.)

$N(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4\bar{a}_5\bar{a}_6\bar{a}_7\bar{a}_8)$

$$= N - S_1 + S_2 - S_3 + S_4 - S_5 + S_6 - S_7 + S_8$$

$N = 8^{10}$

$N(a_1) = 7^{10}$ hence by symmetry $S_1 = \binom{8}{1}7^{10}$

$N(a_1a_2) = 6^{10}$ by symmetry $S_2 = \binom{8}{2}6^{10}$

$N(a_1a_2a_3) = 5^{10}$ " $S_3 = \binom{8}{3}5^{10}$

$N(a_1a_2a_3a_4) = 4^{10}$ " $S_4 = \binom{8}{4}4^{10}$

$N(a_1a_2a_3a_4a_5) = 3^{10}$ $S_5 = \binom{8}{5}3^{10}$

$N(a_1a_2a_3a_4a_5a_6) = 2^{10}$ $S_6 = \binom{8}{6}2^{10}$

$N(a_1a_2a_3a_4a_5a_6a_7) = 1^{10}$ $S_7 = \binom{8}{7}1^{10}$

$N(a_1a_2a_3a_4a_5a_6a_7a_8) = 0^{10} = 0$ $S_8 = 0$

So the number of rolls in which all numbers appear is

$$8^{10} - \binom{8}{1}7^{10} + \binom{8}{2}6^{10} - \binom{8}{3}5^{10} + \binom{8}{4}4^{10} - \binom{8}{5}3^{10} + \binom{8}{6}2^{10} - \binom{8}{7}1^{10}$$

Q4 We consider the following sums:

$$b_1 = a_1$$
$$b_2 = a_1 + a_2$$
$$b_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$b_{10} = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}$$

If one of $b_1, \dots, b_{10}$ is divisible by 10, say b_K, then the desired subset of consecutive numbers whose sum is divisible by 10 is in fact $\{a_1, a_2 \dots, a_K\}$.

Otherwise, there will be a remainder when the numbers $b_1, \dots, b_{10}$ are divided by 10, and those remainders will be one of $1, 2, \dots, 9$. So there are 10 numbers each which have one of 9 remainders, hence two have the same remainder, say b_i and b_j where $j > i$.
So $b_j - b_i$ must be divisible by 10 (because it will have zero remainder when divided by 10)
So $\{a_{i+1}, a_{i+2}, \dots, a_j\}$ is the desired subset of consecutive numbers.

5. We consider the following sets of numbers between 0 and 99:

$\{0\}, \{1, 99\}, \{2, 98\}, \ldots, \{49, 51\}, \{50\}$

There are 51 such sets.

We now assign our 52 numbers into these sets according to their remainder when they are divided by 100. Since there are 52 integers assigned one of 51 groups, there is some group that contains two numbers, say m and n.

If these fall into a set having two elements there are two possibilities, either m and n have the same remainder, or they have different remainders.

If they have the same remainder then $m-n$ is divisible by 100. If they have different remainders, then the remainders sum to 100, hence $m+n$ is divisible by 100.

If m and n are both in the set $\{0\}$ or $\{50\}$ then $m+n$ and $m-n$ are divisible by 100.

6 (a) We consider possible piles of 7 marbles where every marble is red or green or yellow or blue.

Let a_1 be the property that the pile has 4 or more red marbles

a_2 4 or more green

a_3 6 or more yellow

a_4 6 or more blue.

$$N(\bar{a}_1\, \bar{a}_2\, \bar{a}_3\, \bar{a}_4) = N - S_1 + S_2 - S_3 + S_4$$

$$N = \binom{7+4-1}{4-1} = \binom{10}{3}$$

$$S_1 = N(a_1) + N(a_2) + N(a_3) + N(a_4)$$

$$N(a_1) = N(a_2) = \binom{(7-4)+4-1}{4-1} = \binom{6}{3}$$

$$N(a_3) = N(a_4) = \binom{(7-6)+4-1}{4-1} = \binom{4}{3}$$

We note that if we satisfied more than one of these conditions, then we would have more than 7 marbles, so $S_2 = S_3 = S_4 = 0$.

So there are

$\binom{10}{3} - \left[\binom{6}{3} + \binom{4}{3}\right]$ way to have 7 marbles where there are not more than 3 reds, not more than 3 greens, not more than 5 yellows and not more than 5 blues.

6(b) We want to consider here distributions of marbles to 7 people where every marble is red or green or yellow or blue.

Let a_1 be the property that in the distribution 4 or more people have red

a_2 4 or more people have green.

a_3 6 or more people have yellow

a_4 6 or more have blue.

$$N(\bar{a}_1\bar{a}_2\bar{a}_3\bar{a}_4) = N - S_1 + S_2 - S_3 + S_4$$

$$N = 4^7$$

$$S_1 = N(a_1) + N(a_2) + N(a_3) + N(a_4)$$

$$N(a_1) = N(a_2) = \binom{7}{4}3^3 + \binom{7}{5}3^2 + \binom{7}{6}3^1 + \binom{7}{7}3^0$$

{Note; we need to count distributions which have exactly 4 red, exactly 5 red, exactly 6 red and exactly 7 red, and then sum them together. If we tried $\binom{7}{4}4^3$ we would double count some cases.}

$$N(a_3) = N(a_4) = \binom{7}{6}3^1 + \binom{7}{7}3^0$$

So the number of distributions having an appropriate amount of each colour is

$$4^7 - \left(2\binom{7}{4}3^3 + 2\binom{7}{5}3^2 + 4\binom{7}{6}3^1 + 4\binom{7}{7}3^0\right).$$

7.

Let a_1 be the property that a 6 card hand has no diamonds.
" a_2 ... no clubs.
" a_3 ... no hearts.
" a_4 ... no spades.

a) $\binom{52}{6}$ $(= N)$

b) $N(\overline{a_1}\,\overline{a_2}\,\overline{a_3}\,\overline{a_4}) = N - S_1 + S_2 - S_3 + S_4$

$N(a_1) = \binom{39}{6}$ by symmetry $S_1 = \binom{4}{1}\binom{39}{6}$
$N(a_1a_2) = \binom{26}{6}$ " $S_2 = \binom{4}{2}\binom{26}{6}$
$N(a_1a_2a_3) = \binom{13}{6}$ " $S_3 = \binom{4}{3}\binom{13}{6}$
$N(a_1a_2a_3a_4) = 0$ $S_4 = 0$

So the number of 6 card hands having all suits present is
$$\binom{52}{6} - \binom{4}{1}\binom{39}{6} + \binom{4}{2}\binom{26}{6} - \binom{4}{3}\binom{13}{6} + 0$$

c) $e_2 = S_2 - \binom{3}{1}S_3 + \binom{4}{2}S_4$

$$= \binom{4}{2}\binom{26}{6} - \binom{3}{1}\binom{4}{3}\binom{13}{6} + 0$$

The number of 6 card hands having 2 suits is
$$\binom{4}{2}\binom{26}{6} - \binom{3}{1}\binom{4}{3}\binom{13}{6}$$

8. We want to consider paths that go from $(0,0)$ to $(8,6)$.

Let a_1 be the property that the path goes over $(2,2)-(2,3)$.

a_2 $(3,4)-(4,4)$.

a_3 $(6,4)-(6,5)$.

$$N(\bar{a}_1\bar{a}_2\bar{a}_3) = N - S_1 + S_2 - S_3$$

$$N = \binom{14}{6} \left\{\text{or } \binom{14}{8}\right\}$$

$$\left.\begin{aligned} N(a_1) &= \binom{4}{2}\binom{9}{3} \\ N(a_2) &= \binom{7}{4}\binom{6}{2} \\ N(a_3) &= \binom{10}{4}\binom{3}{1} \end{aligned}\right\} S_1 = \binom{4}{2}\binom{9}{3} + \binom{7}{4}\binom{6}{2} + \binom{10}{4}\binom{3}{1}$$

$$\left.\begin{aligned} N(a_1a_2) &= \binom{4}{2}\binom{2}{1}\binom{6}{2} \\ N(a_1a_3) &= \binom{4}{2}\binom{5}{1}\binom{3}{1} \\ N(a_2a_3) &= \binom{7}{4}\binom{2}{0}\binom{3}{1} \end{aligned}\right\} S_2 = \binom{4}{2}\binom{2}{1}\binom{6}{2} + \binom{4}{2}\binom{5}{1}\binom{3}{1} + \binom{7}{4}\binom{2}{0}\binom{3}{1}$$

$$N(a_1a_2a_3) = \binom{4}{2}\binom{2}{1}\binom{2}{0}\binom{3}{1} = S_3$$

So the number of paths that do not cross broken links are

$$\binom{14}{6} - \left(\binom{4}{2}\binom{9}{3} + \binom{7}{4}\binom{6}{2} + \binom{10}{4}\binom{3}{1}\right) + \left(\binom{4}{2}\binom{2}{1}\binom{6}{2} + \binom{4}{2}\binom{5}{1}\binom{3}{1} + \binom{7}{4}\binom{2}{0}\binom{3}{1}\right)$$

$$- \binom{4}{2}\binom{2}{1}\binom{2}{0}\binom{3}{1}.$$