Q1 Since X is of size 4, $X \times X$ has size 16. Of these 16, there are 4 of type (a,a), and the remaining 12 of type (a,b) for $a \neq b$.

a) Every binary relation is a subset of $X \times X$. There are 2^{16} such subsets.

b) For a binary relation to be reflexive, it must contain the 4 elements that are (a,a). Hence we can think of a reflexive relation as a subset of the remaining 12 elements. Hence there are 2^{12}.

c) We consider grouping together the elements (a,b) with (b,a) where $a \neq b$. There are $\binom{4}{2} = 6$ such groups (also note this is $12/2$). Each of these groups is either in the relation or not in the relation. The same is true for each of the 4 (a,a)'s. Hence there are 2^{10} symmetric binary relations.

d) This is not easily countable. It is known to be 3994. The number of transitive relations on X where $|X| = 2$ is the largest set I could expect you to count. (Write out all 16 relations and eliminate the ones that are not transitive.).

(e) For each of the 6 pairs (a,b) and (b,a) where $a \neq b$, either (a,b) is in the relation, (b,a) is in the relation or both are in the relation. For each of the 4 (a,a)'s, they are either in the relation or they are not in the relation. Hence there are $3^6 \cdot 2^4$ complete relations.

(f) By asymmetric, (a,a) is not in the relation. Also by asymmetric, for $a \neq b$ at most one of (a,b) and (b,a) may be in the relation. By complete, at least one of (a,b) and (b,a) is in the relation. Hence there are 2^6 binary relations that are asymmetric and complete.

2. Here, since X is the set of subsets of A, $|X| = 16$ and $|X \times X| = 16 \times 16 = 2^8 = 256$. Of these, there are 16 of type (a,a) and 240 of (a,b) where $a \neq b$.

a) There are 2^{256} binary relations.
b) There are 2^{240} reflexive binary relations.
c) There are 2^{136} symmetric binary relations.
d) There are $3^{120} \cdot 2^{16}$ complete binary relations.
e) There are 2^{120} complete asymmetric binary relations.

3. For a relation to be reflexive we insist that (a,a) is in the relation. Hence we can consider these as a subset of the $n(n-1)$ remaining elements of $X \times X$; Hence there are $2^{n(n-1)}$ reflexive binary relations.
For a relation to be irreflexive we insist that (a,a) is not in the relation. So it is a subset of the remaining $n(n-1)$ elements of $X \times X$; Hence there are also $2^{n(n-1)}$ irreflexive binary relations.

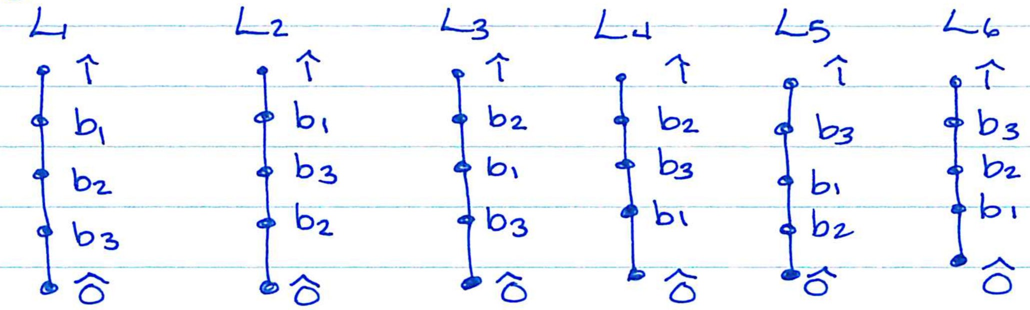
Hence neither is greater.

4. For a symmetric relation, for each of the $\binom{n}{2}$ pairs of (a,b) and (b,a) where $a \neq b$, the pair is either in or not in. For each of the n (a,a)'s, it is in or not in. So there are $2^{\frac{n(n-1)}{2}} \cdot 2^n$ symmetric binary relations on X.

For antisymmetric, for each of the $\binom{n}{2}$ pairs we have 3 choices; (a,b) is in the relation, (b,a) is in the relation or neither is in. ((a,a)'s may be in or not).
So there are $3^{n\frac{(n-1)}{2}} \cdot 2^n$ antisymmetric binary relations

Hence if $n > 1$, there are more antisymmetric relations.

5. For any linear extension, $\hat{1}$ is the maximum and $\hat{0}$ is the minimum; so there are 6 possible ways to order the antichain b_1, b_2, b_3.



Since the partial order was not already a linear order, we know the dimension has to be larger than 1. Since L_1 and L_6 intersect to give the partial order back, we know dimension is at most 2.

Therefore the dimension is 2.

6

P1)

a)

<u>Longest Chain</u>

$\hat{1}$

a_1

a_4

a_8

$\hat{0}$

<u>5 disjoint antichains covering X</u>

$\hat{1}$

$a_1 \quad a_2 \quad a_3$

$a_4 \quad a_5 \quad a_6 \quad a_7$

$a_8 \quad a_9 \quad a_{10}$

$\hat{0}$

b) <u>Largest antichain</u>

$a_4 \quad a_5 \quad a_6 \quad a_7$

<u>4 disjoint chains covering X</u>

$\hat{1}, a_1, a_4, a_8, \hat{0}$

a_2, a_5, a_9

a_3, a_6, a_{10}

a_7

c) The height is 5, the width is 4

$$1 < \dim P_1 \leqslant 4$$

D6

P2) Longest Chain

$\hat{1}$

b_2

b_4

b_6

$\hat{0}$

5 disjoint antichains covering X

$\hat{1}$

$b_1 \; b_2 \; b_3$

$b_4 \; b_5$

$b_6 \; b_7 \; b_8$

$\hat{0}$

Largest Antichain

$b_1 \; b_2 \; b_3$

3 disjoint chains covering X

$\hat{1}$, b_1, b_4, b_6, $\hat{0}$

b_2, b_5, b_7

b_3, b_8

The height is 5, the width is 3

$$1 < \dim P_2 \leq 3$$

6

P3) **Longest Chain**

d_1

d_5

d_9

3 disjoint antichains covering X

$d_1\ d_2\ d_3\ d_4$

$d_5\ d_6\ d_7\ d_8$

$d_9\ d_{10}\ d_{11}\ d_{12}$

Largest Antichain

$d_1 d_2 d_3 d_4$

4 disjoint chains covering X

d_1	d_2	d_3	d_4
d_6	d_5	d_7	d_8
d_{11}	d_9	d_{10}	d_{12}

The height is 3, the width is 4

$1 < \dim \leq 4$