MATH 2030 Worksheet 9 Solutions

Q1

(a) i. A longest chain in P_1 is $\begin{matrix} a_1 \\ b_1 \\ c_1 \end{matrix}$. There are others.

ii. There will be 3 antichains in a set of antichains that contain all elements.
$\{a_1\ a_2\ a_3\}$ $\{b_1\ b_2\ b_3\}$ and $\{c_1\ c_2\ c_3\}$ is such a set.

(b) i. A largest antichain in P_1 is $\{b_1, b_2, b_3, c_2\}$.

ii There will be 4 chains in a set of chains that contain all elements.
$\begin{matrix} a_1 \\ b_1 \\ c_1 \end{matrix}$ $\begin{matrix} a_3 \\ b_3 \\ c_3 \end{matrix}$ $\begin{matrix} a_2 \\ b_2 \end{matrix}$ $\cdot\, c_2$ is an appropriate set of chains

(c) $1 < \dim P_1 \leq 4$
These two linear extensions should intersect in the original partial order, showing the dimension is 2.

(top) $a_1\ \ a_3\ \ c_2\ \ a_2\ \ b_1\ \ b_2\ \ b_3\ \ c_1\ \ c_3$ (bottom)

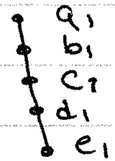
$a_2\ \ a_3\ \ a_1\ \ b_3\ \ b_2\ \ b_1\ \ c_3\ \ c_1\ \ c_2$

(d) P_1 is obviously not a lattice, as pairs such as a_1, a_2 have no join.

Worksheet 9

Q1 (P_2)

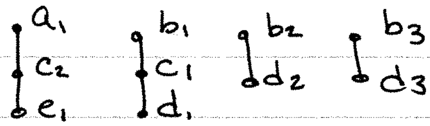
(a) i longest chain



ii $\{a_1\}$, $\{b_1, b_2, b_3\}$, $\{c_1 c_2\}$, $\{d_1 d_2 d_3\}$, $\{e_1\}$

(b) i a largest antichain $\{b_1, b_2, b_3, c_2\}$

ii



(c) $1 < \dim P_2 \leq 4$

The follow two linear extensions should intersect in the original order; showing $\dim P_2 = 2$.

(bottom)

(top) $a_1\ c_2\ b_1\ b_2\ b_3\ c_1\ d_1\ d_2\ d_3\ e_1$

$a_1\ b_3\ b_2\ b_1\ c_1\ d_3\ d_2\ d_1\ c_2\ e_1$

(d) Yes, P_2 is a lattice

Q1 (d) Justification:

a_1 is $\hat{1}$ and e_1 is $\hat{0}$

$\hat{1} \vee x = \hat{1}$ and $\hat{1} \wedge x = x \quad \forall x \in X$
$\hat{0} \vee x = x$ and $\hat{0} \wedge x = \hat{0} \quad \forall x \in X$

$c_2 \vee x = \hat{1}$ and $c_2 \wedge x = \hat{0} \quad \forall x \in X \setminus \{\hat{1}, \hat{0}, c_2\}$

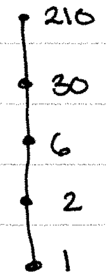
$b_i \vee b_j = \hat{1} \quad b_i \wedge b_j = c_1 \quad (i \neq j)$
$b_i \vee x = b_i \quad b_i \wedge x = x \quad \forall x \in \{c_1, d_1, d_2, d_3\}$

$d_i \vee d_j = c_1 \quad d_i \wedge d_j = \hat{0} \quad (i \neq j)$
$d_i \vee x = x \quad d_i \wedge x = d_i \quad \forall x \in \{c_1, b_1, b_2, b_3\}$

This shows all meets and joins exist.

2 (a)

Chain



antichains

{210}

{30, 42, 70, 105}

{6, 10, 14, 15, 21, 35}

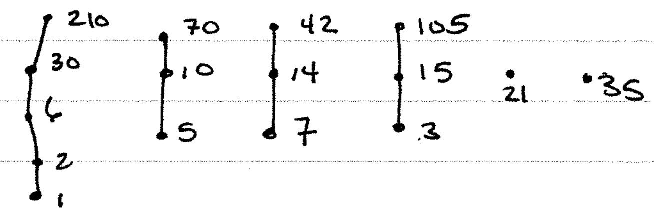
{2, 3, 5, 7}

{1}

(b) antichain

{6, 10, 14, 15, 21, 35}

Chains



(c) $1 < \dim P_3 \leq 6$.

Since this partial order is the same as subsets of a set of 4 elements, the dimension is known to be 4 (Komm [1948], see Q11 p. 272)

(d) $a \vee b$ is the least common multiple of a & b.

$a \wedge b$ is the greatest common divisor.

Since the partial order contains all positive divisors of 210, least common multiples and greatest common divisors are in the partial order as well, hence it is a lattice.

e) For any $x \in X$, the complement of x is $^{210}/x$.

f) Since the lattice is complemented, we only need to show whether or not it is a distributive lattice.

We can describe every element of this lattice uniquely by the set of primes that divide it. (The set of primes will be a subset of 2, 3, 5 and 7).

Suppose a prime divides $a \wedge (b \vee c)$, then it divides a and it divides either b or c (or both). Then that prime divides either a and b or a and c, hence it will divide $(a \wedge b) \vee (a \wedge c)$.

This shows $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, and hence the lattice is distributive.

This is a boolean lattice.