Q1 (a) $\dfrac{21!}{(21-10)!} = \dfrac{21!}{11!}$ (b) 3^{10}

(c) $\dbinom{10+3-1}{3-1} = \dfrac{12!}{10!\,2!}$ (d) $\dbinom{21}{10} = \dfrac{21!}{11!\,10!}$

(d) Note: first we choose two places for metal beads, then we permute two beads into those places.

$$\binom{12}{2}\left[\frac{21!}{(21-2)!}\right] = \frac{12!\ 21!}{2!\,10!\ 19!}$$

(e) Note: Similar to (d), but there are only 2 metal beads.

$$\binom{12}{2}\left[\frac{2!}{(2-2)!}\right] = \frac{2\cdot 12!}{2!\ 10!} = \frac{12!}{10!}$$

Q2

(a) $\dfrac{20!}{(20-3)!} = \dfrac{20!}{17!}$ (OR) $20\cdot 19\cdot 18$

(b) 3^{20}

(c) $\dbinom{20+3-1}{3-1} = \dbinom{22}{2} = \dfrac{22!}{2!\,20!}$

3 (a) The number of ways to throw five dice is 6^5.
The number of ways they could all be the same is 6.
Hence probability is $\frac{6}{6^5} = \frac{1}{6^4}$

(b) The number of ways five dice could be rolled to include a 1 would be the same as the total number of ways they could be rolled <u>minus</u> the number of ways they could be rolled without a 1.

Hence probability is $\frac{6^5 - 5^5}{6^5}$.

(c) Similar to (b), the number of ways to pick a 7 card hand with an ace is the same as the number of ways to pick a 7 card hand minus the number of ways to pick a 7 card hand without an ace.
The number of 7-card hands is $\binom{52}{7} = \frac{52!}{7!\,45!}$
The number of 7-card hands without an ace is $\binom{48}{7} = \frac{48!}{7!\,41!}$

Hence probability is $\frac{\binom{52}{7} - \binom{48}{7}}{\binom{52}{7}} = \frac{\frac{52!}{45!} - \frac{48!}{41!}}{52!/45!}$.

(d) The number of ways 5 cards can be drawn is $\binom{52}{5}$. The number of ways to choose 5 cards so that there is <u>no</u> pair is $\binom{13}{5}\cdot 4^5$ (First choose the 5 distinct numbers, then assign suits.)

Hence the probability is $\dfrac{\binom{52}{5} - \binom{13}{5}\cdot 4^5}{\binom{52}{5}}$

$$= \frac{52!/5!\,47! - 13!\cdot 4^5/5!\,7!}{52!/5!\,47!} = \frac{52!/47! - 13!\cdot 4^5/7!}{52!/47!}$$

(e) This bag now contains 40 beads. There are then $\binom{40}{3}$ ways to pick 3 beads.
There are $\binom{12}{3}$ ways to pick 3 blue beads, $\binom{15}{3}$ ways to pick 3 yellow beads and $\binom{13}{3}$ ways to pick 3 red beads. These are mutually exclusive, hence there are $\binom{12}{3}+\binom{15}{3}+\binom{13}{3}$ ways to pick 3 beads of the same colour.

Hence the probability is $\dfrac{\binom{12}{3}+\binom{15}{3}+\binom{13}{3}}{\binom{40}{3}}$

$$= \frac{\frac{12!}{9!} + \frac{15!}{12!} + \frac{13!}{10!}}{\frac{40!}{37!}}$$

4(a) [a 3-combination of an 8-set ; order does not matter, no replacements.]
example: Albert has 8 books he wants to read. How many ways can he choose 3 books to bring with him for the weekend at the lake?

(b) [a 3-permutation from an 8-set ; order does matter ; no replacements.]
example: Betty has 8 books. How many ways can she arrange 3 of them on her mantle between bookends?

(c) [a 3-permutation of an 8-set with replacement ; order matters ; replication allowed.]
example: Chris and his two friends go to a book convention and a publisher is there to promote is top 8 bestsellers. He has offered to give Chris and his friends each a book of their choosing. How many different choices are possible.

(d) [an 8-combination of a 3-set with replacement ; order does not matter, replication allowed]
example: Denise is at a book convention and a publisher with 3 different kinds of bookmarks has invited her to take 8 bookmarks. How many ways can this be done?

5. $\sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$

Let suppose we have two n-sets, A and B. Together they have $2n$ elements. Now lets suppose we want to choose n elements from this $2n$-set. We could:

pick 0 elements of A and n elements of B in $\binom{n}{0}\binom{n}{n}$ ways.
pick 1 element of A and $n-1$ elements of B in $\binom{n}{1}\binom{n}{n-1}$ ways.
pick 2 elements of A and $n-2$ elements of B in $\binom{n}{2}\binom{n}{n-2}$ ways

$\vdots$

pick n elements of A and 0 elements of B in $\binom{n}{n}\binom{n}{0}$ ways

Since these cases are mutually exclusive, we can sum them to find the number of ways of picking an n-set from a $2n$ set.

So $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$

or $\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$