MATH 1210 Assignment #1

Solutions

1. Use mathematical induction to prove

   $1 + 5 + 9 + 13 + \cdots + (4n + 1) = (n + 1)(2n + 1)$ for all $n \geq 1$.

Solution:

Let P_n be the statement $1 + 5 + 9 + 13 + \cdots + (4n + 1) = (n + 1)(2n + 1)$.

If $n = 1$ then, since $4(1) + 1 = 5$ we have
$1 + 5 = 6$ and
$((1) + 1)(2(1) + 1) = 2 \cdot 3 = 6$

Hence P_1 is true.

Suppose that for some integer $k \geq 1$ P_k were true. IE suppose
$1 + 5 + 9 + 13 + \cdots + (4k + 1) = (k + 1)(2k + 1)$ is true

(We now want to show that P_{k+1} is also true. IE we want to show
$1 + 5 + 9 + 13 + \cdots + (4(k + 1) + 1) = ((k + 1) + 1)(2(k + 1) + 1))$.
Now

$$\begin{aligned}
&1 + 5 + 9 + 13 + \cdots + (4(k + 1) + 1) \\
=&1 + 5 + 9 + 13 + \cdots + (4k + 5) \\
=&1 + 5 + 9 + 13 + \cdots + (4k + 1) + (4k + 5) \\
=&(k + 1)(2k + 1) + (4k + 5) \\
=&2k^2 + 3k + 1 + (4k + 5) \\
=&2k^2 + 7k + 6 \\
=&(k + 2)(2k + 3) \\
=&((k + 1) + 1)(2(k + 1) + 1)
\end{aligned}$$

Showing that P_{k+1} is true whenever P_k is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, by the Principle of Mathematical Induction, P_n is true for all $n \geq 1$.

2. Use mathematical induction to prove

$$2+5+8+11+\cdots+(9n-1)=\frac{3n(9n+1)}{2} \text{ for all } n \geq 1.$$

Solution:

Let P_n be the statement $2+5+8+11+\cdots+(9n-1)=\dfrac{3n(9n+1)}{2}$.

If $n=1$ then, since $9(1)-1=8$ we have
$2+5+8=15$ and
$\frac{3(1)(9(1)+1)}{2}=\frac{30}{2}=15$
Hence P_1 is true.

Suppose that for some integer $k \geq 1$ P_k were true. IE suppose
$2+5+8+11+\cdots+(9k-1)=\frac{3k(9k+1)}{2}$ is true
(We now want to show that P_{k+1} is also true. IE we want to show
$2+5+8+11+\cdots+(9(k+1)-1)=\frac{3(k+1)(9(k+1)+1)}{2}$).
Now

$$\begin{aligned}
&2+5+8+11+\cdots+(9(k+1)-1)\\
=&2+5+8+11+\cdots+(9k+8)\\
=&2+5+8+11+\cdots+(9k-1)+(9k+2)+(9k+5)+(9k+8)\\
=&\frac{3k(9k+1)}{2}+(9k+2)+(9k+5)+(9k+8)\\
=&\frac{3k(9k+1)}{2}+27k+15\\
=&\frac{27k^2+3k}{2}+\frac{54k+30}{2}\\
=&\frac{27k^2+57k+30}{2}\\
=&\frac{(3k+3)(9k+10)}{2}\\
=&\frac{3(k+1)(9(k+1)+1)}{2}
\end{aligned}$$

Showing that P_{k+1} is true whenever P_k is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, by the Principle of Mathematical Induction, P_n is true for all $n \geq 1$.

3. Use mathematical induction to prove

$n + (n+1) + (n+2) + (n+3) + \cdots + (5n) = 3n(4n+1)$ for all $n \geq 1$.

Solution:

Let P_n be the statement $n + (n+1) + (n+2) + (n+3) + \cdots + (5n) = 3n(4n+1)$.

If $n = 1$ then, since $5(1) = 5$ we have
$1 + 2 + 3 + 4 + 5 = 15$ and

$3(1)(4(1) + 1) = 3 \cdot 5 = 15$

Hence P_1 is true.

Suppose that for some integer $k \geq 1$ P_k were true. IE suppose

$k + (k+1) + (k+2) + (k+3) + \cdots + (5k) = 3k(4k+1)$ is true

(We now want to show that P_{k+1} is also true. IE we want to show
$(k+1)+((k+1)+1)+((k+1)+2)+((k+1)+3)+\cdots+(5(k+1)) = 3(k+1)(4(k+1)+1)$).
Now

$$\begin{aligned}
&(k+1) + ((k+1)+1) + ((k+1)+2) + ((k+1)+3) + \cdots + (5(k+1)) \\
&= (k+1) + (k+2) + (k+3) + (k+4) + \cdots + (5k+5) \\
&= (k+1) + (k+2) + (k+3) + (k+4) + \cdots + (5k) \\
&\qquad\qquad + (5k+1) + (5k+2) + (5k+3) + (5k+4) + (5k+5) \\
&= k + (k+1) + (k+2) + (k+3) + (k+4) + \cdots + (5k) \\
&\qquad\qquad + (5k+1) + (5k+2) + (5k+3) + (5k+4) + (5k+5) - k \\
&= 3k(4k+1) + (5k+1) + (5k+2) + (5k+3) + (5k+4) + (5k+5) - k \\
&= 12k^2 + 3k + 24k + 15 \\
&= 12k^2 + 27k + 15 \\
&= (3k+3)(4k+5) \\
&= 3(k+1)(4(k+1)+1)
\end{aligned}$$

Showing that P_{k+1} is true whenever P_k is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, by the Principle of Mathematical Induction, P_n is true for all $n \geq 1$.

4. Use mathematical induction to prove

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}.$$

Solution:

Let P_n be the statement $1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$.

If $n = 1$ then, since $2(1) = 2$ we have

$1^2 + 2^2 = 1 + 4 = 5$ and

$\frac{(1)(2(1)+1)(4(1)+1)}{3} = \frac{(3)(5)}{3} = 5$

Hence P_1 is true.

Suppose that for some integer $k \geq 1$ P_k were true. IE suppose

$1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}$ is true

(We now want to show that P_{k+1} is also true. IE we want to show

$1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2(k+1))^2 = \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}$).

Now

$$\begin{aligned}
&1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2(k+1))^2 \\
=& 1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2k+2)^2 \\
=& 1^2 + 2^2 + 3^2 + 4^2 + \cdots + (2k)^2 + (2k+1)^2 + (2k+2)^2 \\
=& \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + (2k+2)^2 \\
=& \frac{8k^3 + 6k^2 + k}{3} + \frac{3(4k^2 + 4k + 1)}{3} + \frac{3(4k^2 + 8k + 4)}{3} \\
=& \frac{8k^3 + 30k^2 + 37k + 15}{3} \\
=& \frac{(k+1)(8k^2 + 22k + 15)}{3} \\
=& \frac{(k+1)(2k+3)(4k+5)}{3} \\
=& \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}
\end{aligned}$$

Showing that P_{k+1} is true whenever P_k is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, by the Principle of Mathematical Induction, P_n is true for all $n \geq 1$.

5. Use mathematical induction to prove

$3^{3n} - 1$ is divisible by 13 for all $n \geq 1$.

Solution:

Let P_n be the statement $3^{3n} - 1$ is divisible by 13.

If $n = 1$ then
$3^{3(1)} - 1 = 3^3 - 1 = 27 - 1 = 26$ and
$26 = (13)(2)$, so 26 is divisible by 13.

Hence P_1 is true.

Suppose that for some integer $k \geq 1$ P_k were true. IE suppose

$3^{3k} - 1$ is actually divisible by 13.

(We now want to show that P_{k+1} is also true. IE we want to show
$3^{3(k+1)} - 1$ is divisible by 13).
Now

$$\begin{aligned}
&3^{3(k+1)} - 1 \\
=\,&3^{3k+3} - 1 \\
=\,&3^3 \cdot 3^{3k} - 1 \\
=\,&3^3 \cdot 3^{3k} - 3^3 + 3^3 - 1 \\
=\,&3^3(3^{3k} - 1) + 3^3 - 1 \\
=\,&3^3(3^{3k} - 1) + 26
\end{aligned}$$

Since $3^{3k} - 1$ is divisible by 13 (by assumption), and 26 is divisible by 13 (base case) we know that $3^3(3^{3k} - 1) + 26$ is divisible by 13. This shows that P_{k+1} is true whenever P_k is true.

Since P_1 is true and P_{k+1} is true whenever P_k is true, by the Principle of Mathematical Induction, P_n is true for all $n \geq 1$.

6. Write each of the following using sigma notation:

(a) $1+3+5+7+\cdots+111$

Solution:

$$1+3+5+7+\cdots+111=\sum_{j=1}^{56}(2j-1)$$

(b) $\frac{5}{12}+\frac{6}{14}+\frac{7}{16}+\frac{8}{18}+\cdots+\frac{49}{100}$

Solution:

$$\frac{5}{12}+\frac{6}{14}+\frac{7}{16}+\frac{8}{18}+\cdots+\frac{49}{100}=\sum_{j=1}^{45}\frac{j+4}{2j+10}=\sum_{k=5}^{49}\frac{k}{2k+2}$$

(c) $2+6+10+14+\cdots+(8n+6)$

Solution:

$$2+6+10+14+\cdots+(8n+6)=\sum_{j=1}^{2n+2}(4j-2)$$

7. Using the known formulas, evaluate each of the following:

(a) $\sum_{j=1}^{12}(j+1)^2$

Solution:

$\sum_{j=1}^{12}(j+1)^2$

$=\sum_{j=1}^{12}(j^2+2j+1)$

$=\sum_{j=1}^{12}j^2+2\sum_{j=1}^{12}j+\sum_{j=1}^{12}1$

$=\frac{(12)(13)(25)}{6}+2\frac{(12)(13)}{2}+(12)$

$=650+156+12=818$

(b) $\sum_{j=6}^{20}4j-7$

Solution:

$\sum_{j=6}^{20}4j\ -7$

$=\sum_{k=1}^{15}4(k+5)\ -7$

$=\sum_{k=1}^{15}4k+20\ -7$

$= 4\sum_{k=1}^{15} k + 20\sum_{k=1}^{15} 1 \ - 7$

$= 4\frac{(15)(16)}{2} + (20)(15) \ - 7$

$= 480 + 300 - 7 = 773$

This was the intended (but incorrect) solution:
$\sum_{j=6}^{20}(4j-7)$

$= \sum_{k=1}^{15}(4k+13)$

$= 4\sum_{k=1}^{15} k + 13\sum_{k=1}^{15} 1$

$= 4\frac{(15)(16)}{2} + (13)(15)$

$= 480 + 195 = 675$

(c) $\sum_{j=6}^{20}(j-5)(j+3)$

Solution:
$\sum_{j=6}^{20}(j-5)(j+3)$

$= \sum_{k=1}^{15}(k)(k+8)$

$= \sum_{k=1}^{15} k^2 + 8\sum_{k=1}^{15} k$

$= \frac{(15)(16)(31)}{6} + 8\frac{(15)(16)}{2}$

$= 1240 + 960 = 2200$