MATH 1210 Assignment #2
Solutions

Due: February 3, 2016; At the start of class

Reminder: all assignments *must* be accompanied by a signed copy of the honesty declaration available on the course website.

1. Simplify and express the complex numbers in Cartesian form

   (a) $\overline{\left(\frac{(6-2i)^4}{(1+3i)^4}\right)}$

   (b) $\frac{(i-1)^3}{(i+1)^2}$

   (c) $\left(\frac{i}{e^{i\pi}}\right)^{25}$

Solution:

(a) $\overline{\left(\frac{(6-2i)^4}{(1+3i)^4}\right)} = \overline{\left(\frac{6-2i}{1+3i}\right)^4} = \overline{\left(\frac{(6-2i)(1-3i)}{(1+3i)(1-3i)}\right)^4} = \overline{\left(\frac{6-18i-2i-6}{1+9}\right)^4} = \overline{\left(\frac{-20i}{10}\right)^4} = \overline{2^4(i^2)^2} = \overline{16} = 16$

(b) $\frac{(i-1)^3}{(i+1)^2} = \frac{(\sqrt{2}(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i))^3}{(\sqrt{2}(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}))^2} = \frac{(\sqrt{2}e^{i3\pi/4})^3}{(\sqrt{2}e^{i\pi/4})^2} = \sqrt{2}e^{i(9\pi/4-2\pi/4)} = \sqrt{2}e^{i(7\pi/4)} = \sqrt{2}(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}) = 1 - i$

(c) $\left(\frac{i}{e^{i\pi}}\right)^{25} = \left(\frac{e^{i\pi/2}}{e^{i\pi}}\right)^{25} = \left(e^{i(\pi/2-\pi)}\right)^{25} = \left(e^{-i\pi/2}\right)^{25} = (-i)^{25} = -(i^2)^{12}i = -i$

2. Simplify and express the complex numbers in polar and exponential forms using the principal value of the argument θ, $\theta \in (-\pi, \pi]$

   (a) $\left(\overline{\sqrt{3}+3i}\right)^2$

   (b) $-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$

   (c) $(-12+i)^3(-12-i)^3$

Solution:

(a)

$$\left(\overline{\sqrt{3}+3i}\right)^2 = \left(\sqrt{3}-3i\right)^2 = -6 - i6\sqrt{3} = 6(-1-i\sqrt{3}) = 12(-\frac{1}{2} - \frac{i\sqrt{3}}{2})$$

Polar form: $12(\cos(-2\pi/3) + i\sin(-2\pi/3))$
Exponential form: $12e^{-i2\pi/3}$

(b)

$$-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

Polar form: $\cos(-3\pi/4) + i\sin(-3\pi/4)$
Exponential form: $e^{-i3\pi/4}$

(c)

$$(-12+i)^3(-12-i)^3 = ((-12+i)(-12-i))^3 = ((-12)^2+1^2)^3 = 145^3$$

Polar form: $145^3\,(\cos(0) + i\sin(0))$
Exponential form: 145^3e^{i0}

3. Find all solutions of the equation
$x^6 + x^3 + 1 = 0.$

Solution: The polynomial is a polynomial of degree 6. From the Fundamental Theorem of Algebra II, there are 6 solutions to this polynomial equation.

Consider $u = x^3$. The polynomial equation

$$x^6 + x^3 + 1 = (x^3)^2 + (x^3)^1 + 1 = 0$$

can be rewritten as follows

$$u^2 + u + 1 = 0.$$

Roots of the quadratic equation are $u_{1,2} = \frac{-1\pm i\sqrt{3}}{2}$.

Now, we have to find the cubic roots of $\frac{-1\pm i\sqrt{3}}{2}$. Solve

$$x^3 = \frac{-1-i\sqrt{3}}{2},$$

and

$$x^3 = \frac{-1+i\sqrt{3}}{2}.$$

First, we solve

$$x^3 = \frac{-1 - i\sqrt{3}}{2} = e^{i(-2\pi/3+2k\pi)} \Rightarrow (x^3)^{1/3} = \left(e^{i(-2\pi/3+2k\pi)}\right)^{1/3}$$

with $k = 0, 1, 2$. We obtain 3 roots, equally spaced on the circle of radius 1, with an angle $2\pi/3$ between successive roots; the first root x_0 has an argument $-2\pi/9$. The roots are then

$$\begin{aligned} x_0 =& e^{-i2\pi/9}, \\ x_1 =& e^{i4\pi/9}, \\ x_2 =& e^{i10\pi/9} \quad \text{or with the principal value } x_2 = e^{-8\pi/9}. \end{aligned}$$

Second, we solve

$$x^3 = \frac{-1 + i\sqrt{3}}{2} = e^{i(2\pi/3+2k\pi)} \Rightarrow (x^3)^{1/3} = \left(e^{i(2\pi/3+2k\pi)}\right)^{1/3}$$

with $k = 0, 1, 2$. We obtain 3 roots, equally spaced on the circle of radius 1, with an angle $2\pi/3$ between successive roots; the first root x_3 has an argument $2\pi/9$. The roots are then

$$\begin{aligned} x_3 =& e^{i2\pi/9}, \\ x_4 =& e^{i8\pi/9}, \\ x_5 =& e^{i14\pi/9} \quad \text{or with the principal value } x_5 = e^{-4\pi/9}. \end{aligned}$$

The 6 solutions are x_i with $i \in \{0, 1, 2, 3, 4, 5\}$.

4. Find all solutions of the equation

   $z^8 = -1$.

Solution: Find the 8^{th} roots of -1:

$$z^8 = -1 = e^{i(-\pi+2k\pi)} \Rightarrow (z^8)^{1/8} = \left(e^{i(-\pi+2k\pi)}\right)^{1/8} = e^{i(-\pi/8+k\pi/4)}$$

with $k = 0, 1, 2, 3, 4, 5, 6, 7$.

The 8 solutions are equally spaced on the circle of radius 1, with an angle $\pi/4$ between successive roots; the first root has an argument $-\pi/8$. The solutions are

$$\begin{aligned}
z_0 =& e^{-i\pi/8}, \\
z_1 =& e^{i\pi/8}, \\
z_2 =& e^{i3\pi/8}, \\
z_3 =& e^{i5\pi/8}, \\
z_4 =& e^{i7\pi/8}, \\
z_5 =& e^{i9\pi/8}, \quad \text{or with the principal value } z_5 = e^{-7\pi/8}, \\
z_6 =& e^{i11\pi/8}, \quad \text{or with the principal value } z_6 = e^{-5\pi/8}, \\
z_7 =& e^{i13\pi/8} \quad \text{or with the principal value } z_7 = e^{-3\pi/8}.
\end{aligned}$$

5. Let z_1 and z_2 be 2 complex numbers. Show that

   $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$.

Solution: Define $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Their sum is

$$z_1 + z_2 = a_1 + a_2 + i(b_1 + b_2).$$

Take the conjugate on both sides:

$$\begin{aligned}
\overline{z_1 + z_2} &= \overline{a_1 + a_2 + i(b_1 + b_2)} \\
&= a_1 + a_2 - i(b_1 + b_2) \\
&= a_1 - ib_1 + a_2 - ib_2 \\
&= \overline{z_1} + \overline{z_2}.
\end{aligned}$$

6. Let z be a complex number. Using mathematical induction prove that

   $\overline{z^n} = \overline{z}^n$, for all $n \geq 1$.

Solution: Preliminary result: Define two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$; multiply:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

Take the conjugate on both sides:

$$\begin{aligned}
\overline{z_1 z_2} &= \overline{r_1 r_2 e^{i(\theta_1 + \theta_2)}} \\
&= r_1 r_2 e^{-i(\theta_1 + \theta_2)} = r_1 r_2 e^{-i\theta_1} e^{-i\theta_2} = r_1 e^{-i\theta_1} r_2 e^{-i\theta_2} = \overline{z_1}\ \overline{z_2}.
\end{aligned}$$

We want to prove that $P_n : \overline{z^n} = \overline{z}^n$, for all $n \geq 1$.

1. If $n = 1$, we have $\overline{z^1} = \overline{z} = \overline{z}^1$. Therefore P_1 is true.
2. Assume that $P_k : \overline{z^k} = \overline{z}^k$ is true for some $k \geq 1$.

   When $n = k + 1$,

$$\begin{aligned}
\overline{z^{k+1}} &= \overline{z^k z} \\
&= \overline{z^k}\,\overline{z} && [\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \quad z_1, z_2 \in \mathbb{C}] \\
&= \overline{z}^k \overline{z} && [\overline{z^k} = \overline{z}^k, \text{use } P_k] \\
&= \overline{z}^{k+1}
\end{aligned}$$

   We have proved that P_{k+1} is true when P_k is true.
3. By the Principle of Mathematical Induction, we can conclude that P_n is a true proposition for all $n \geq 1$.

7. Consider the following polynomial $P(x) = x^5 - 2x^4 + 4x^3 + 2x^2 - 5x$.
   (a) Verify that $1 + 2i$ is a root of $P(x) = 0$.
   (b) Find all the roots of $P(x) = 0$.
   (c) Factor $P(x)$ into the product of real linear and irreducible real quadratic factors.

Solution:

(a) $P(1 + 2i) = 0$ therefore $1 + 2i$ is a root.

(b) $P(x)$ is a polynomial of degree 5. By the Fundamental Theorem of Algebra II, P has exactly 5 roots (counting multiplicities).

As P has real coefficients, if z is a complex root then $\overline{z}$ is also a root of P. Therefore, as $1 + 2i$ is a root, $1 - 2i$ is also a root of P.

Moreover, x can be factored as

$$P(x) = x(x^4 - 2x^3 + 4x^2 + 2x - 5),$$

so 0 is also a root. So far, we have:

$$P(x) = x(x - 1 - 2i)(x - 1 + 2i)Q_2(x) = x(x^2 - 2x + 5)Q_2(x)$$

where $Q_2(x)$ is a polynomial of degree 2. To find the last 2 roots, we first need to find $Q_2(x)$. Performing for instance the long division of $x^4 - 2x^3 + 4x^2 + 2x - 5$ by $x^2 - 2x + 5$ gives

$$x^4 - 2x^3 + 4x^2 + 2x - 5 = (x^2 - 2x + 5)(x^2 - 1),$$

where $Q_2(x) = x^2 - 1 = (x-1)(x+1)$.

Otherwise, to find the last 2 roots, we could have noticed that $P(1) = 0$ and $P(-1) = 0$.

Summing up, roots of P are 0, $1 \pm 2i$ and ± 1.

(c) So $P(x)$ is factored into a product of real linear and irreducible real quadratic factors as

$$P(x) = x(x-1)(x+1)(x^2 - 2x + 5)$$

8. (a) Show that $(x - i)$ and $(x - 1)$ are linear factors of

$x^4 - 2(1+i)x^3 + 4ix^2 + 2(1-i)x - 1 = 0$.

(b) Factor the polynomial $x^4 - 2(1+i)x^3 + 4ix^2 + 2(1-i)x - 1$ in linear factors.

Solution:

(a) $P(i) = i^4 - 2(1+i)i^3 + 4ii^2 + 2(1-i)i - 1 = 1 + 2(1+i)i - 4i + 2(1-i)i - 1 = 0$. By the Factor Theorem, as $P(i) = 0$, $(x - i)$ is a linear factor of P.
$P(1) = 1 - 2(1+i) + 4i + 2(1-i) - 1 = 0$. By the Factor Theorem, as $P(1) = 0$, $(x-1)$ is a linear factor of P.

(b) Factor the polynomial: From (a)

$$P(x) = x^4 - 2(1+i)x^3 + 4ix^2 + 2(1-i)x - 1 = (x-i)(x-1)Q_2(x)$$

where $Q_2(x)$ is a polynomial of degree 2 that can be found by long division or by identification.

By identification: Assume that $Q_2(x) = ax^2 + bx + c$, then

$$P(x) = (x^2 - (1+i)x + i)(ax^2 + bx + c)$$

where $x^2 - (1+i)x + i = (x-i)(x-1)$. Expand

$$(x^2 - (1+i)x + i)(ax^2 + bx + c)$$

and identify the coefficients of the terms of same degree:

$$\begin{aligned} P(x) =& x^4 - 2(1+i)x^3 + 4ix^2 + 2(1-i)x - 1 \\ =& ax^4 + bx^3 + cx^2 - a(1+i)x^3 - b(1+i)x^2 - c(1+i)x + aix^2 + bix + ci. \end{aligned}$$

- Terms of degree $n = 4$: $1 = a$.
- Terms of degree $n = 3$: $-2(1+i) = b - a(1+i)$.

- Terms of degree $n = 2$: $4i = c - b(1 + i) + ai$.
- Terms of degree $n = 1$: $2(1 - i) = -c(1 + i) + bi$.
- Terms of degree $n = 0$: $-1 = ci$.

We obtain $c = i$, $a = 1$ and $b = -(1 + i)$. Finally,

$$P(x) = (x - i)(x - 1)(x^2 - (1 + i)x + i) = (x - i)(x - 1)(x - i)(x - 1).$$

So $P(x)$ has 2 linear factors $(x - i)$ and $(x - 1)$ of multiplicity 2.

9. Consider the following polynomial
$P(x) = x^5 - 11x^4 + 43x^3 - 73x^2 + 56x - 16$.
   (a) Show that $P(x)$ can be rewritten as $P(x) = Q(x)(x - 4)$ and $P(x) = T(x)(x - 1)$ where $Q(x)$ and $T(x)$ are polynomials in x. Give the degree of $Q(x)$ and $T(x)$.
   (b) Show that 4 is a root of multiplicity 2 of $P(x)$.
   (c) Factor $P(x)$.

Solution:

(a) $P(4) = 0$, so by the Factor Theorem, $(x-4)$ is a linear factor of $P(x)$. Therefore, we can write $P(x) = (x - 4)Q(x)$, where $Q(x)$ is a polynomial of degree 4.

$P(1) = 0$, so by the Factor Theorem, $(x-1)$ is a linear factor of $P(x)$. Therefore, we can write $P(x) = (x - 1)T(x)$, where $T(x)$ is a polynomial of degree 4.

(b) $P(x)$ can be rewritten as

$$P(x) = (x - 1)(x - 4)Q_3(x) = (x^2 - 5x + 4)Q_3(x)$$

where $Q_3(x)$ is a polynomial of degree 3. To find $Q_3(x)$, perform long division or identification of like parameters as in Question 8. We find

$$Q_3(x) = x^3 - 6x^2 + 9x - 4.$$

As $Q_3(4) = 0$, $(x - 4)$ is a linear factor of $Q_3(x)$ and so $(x - 4)$ appears for a second time in the factorization of $P(x)$:

$$P(x) = (x - 1)(x - 4)(x^3 - 6x^2 + 9x - 4) = (x - 1)(x - 4)(x - 4)Q_2(x)$$

where $Q_2(x)$ is a polynomial of degree 2 that we can obtain by dividing $Q_3(x) = x^3 - 6x^2 + 9x - 4$ by $(x - 4)$. The result of the long division of $Q_3(x) = x^3 - 6x^2 + 9x - 4$ by $(x - 4)$ gives $Q_2(x) = x^2 - 2x + 1 = (x - 1)^2$. $x - 4$ is not a factor of $Q_2(x)$. The factor $(x - 4)$ appears only 2 times in the factorization of P, therefore the root $x = 4$ has multiplicity 2.

(c) So $P(x)$ factors as

$$P(x) = (x-1)^3(x-4)^2$$