MATH 1210 Assignment #4

Due: March 16, 2016; At the start of class

Reminder: all assignments *must* be accompanied by a signed copy of the honesty declaration available on the course website.

1. Find the line through $(3,1,-2)$ that intersects and is perpendicular to the line $x=-1+t$, $y=-2+t$, $z=-1+t$ with $t \in \mathbb{R}$.

Solution: To find the equation of the line L we need a point and a direction vector. We have a point $(3,1,-2)$; we have to find a direction vector $\mathbf{v}$. The line L is perpendicular to the line L_1, which is passing through the point $(-1,-2,-1)$ and having a direction vector $\langle 1,1,1\rangle$, so $\mathbf{v}$ is perpendicular to $\langle 1,1,1\rangle$. The line L passes through $(3,1,-2)$ and intersects the line L_1 at a point P, so P is on L and also on L_1. So P satisfies $x=-1+t$, $y=-2+t$, $z=-1+t$ with $t\in\mathbb{R}$; we need to find t corresponding to this point.

The vector starting at $(3,1,-2)$ and finishing at P is along the line L, so it is a direction vector for L: $\mathbf{v}=\langle 3-(-1+t), 1-(-2+t), -2-(-1+t)\rangle = \langle 4-t, 3-t, -1-t\rangle$. As $\mathbf{v}$ is perpendicular to $\langle 1,1,1\rangle$, we have

$$\langle 4-t, 3-t, -1-t\rangle \cdot \langle 1,1,1\rangle = 4-t+3-t-1-t = 6-3t = 0$$

so $t=2$. Therefore, $\mathbf{v}=\langle 4-2, 3-2, -1-2\rangle = \langle 2,1,-3\rangle$. Using $(3,1,-2)$ and $\mathbf{v}=\langle 2,1,-3\rangle$ the parametric equations for L are

$$\begin{aligned} x =& 3+2t, \\ y =& 1+1t, \\ z =& -2-3t, \end{aligned}$$

and the symmetric equations are

$$\frac{x-3}{2} = y-1 = \frac{-z-2}{3}.$$

2. Find an equation for the plane containing the two lines of equations

$$\langle x,y,z\rangle = \langle 0,1,2\rangle + t_1\langle 2,3,-1\rangle, \quad t_1\in\mathbb{R}$$

and

$$\langle x,y,z\rangle = \langle 2,-1,0\rangle + t_2\langle 4,6,-2\rangle, \quad t_2\in\mathbb{R}.$$

Solution: To find the equation of a plane we need a point in the plane and its normal vector. We have 2 lines: L_1 passing through the point $(0, 1, 2)$ and having as direction vector $\mathbf{v_1} = \langle 2, 3, -1\rangle$ and L_2 passing through the point $(2, -1, 0)$ and having as direction vector $\mathbf{v_2} = \langle 4, 6, -2\rangle$. We note that L_1 and L_2 are parallel as their direction vectors are in the same direction, $2\mathbf{v_1} = \mathbf{v_2}$.

If we have 2 (non-parallel) vectors belonging to the plane by using their cross product we will obtain the vector normal to the plane. We have already one vector $\mathbf{v_1}$ (or $\mathbf{v_2}$); as a second vector we consider the vector starting from $(0, 1, 2)$ (point of L_1) and finishing at $(2, -1, 0)$ (point of L_2). This vector $\langle 2 - 0, -1 - 1, 0 - 2\rangle = \langle 2, -2, -2\rangle$ belongs to the plane as the 2 points belong to the plane. To find the normal vector to the plane, compute the cross product of $\langle 2, -2, -2\rangle$ and $\mathbf{v_1}$ (or $\mathbf{v_2}$):

$$\begin{aligned}\langle 2, -2, -2\rangle \times \langle 2, 3, -1\rangle &= \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ 2 & -2 & -2 \\ 2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} -2 & -2 \\ 3 & -1 \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} 2 & -2 \\ 2 & -1 \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} 2 & -2 \\ 2 & 3 \end{vmatrix} \vec{\mathbf{k}} \\ &= 8\vec{\mathbf{i}} - 2\vec{\mathbf{j}} + 10\vec{\mathbf{k}}\end{aligned}$$

For the equation of the plane, we use the point $(2, -1, 0)$ and the normal vector $\langle 8, -2, 10\rangle$ and obtain

$$8(x - 2) - 2(y + 1) + 10z = 0.$$

3. Show that the lines $x - 3 = 4t_1$, $y - 4 = t_1$, $z - 1 = 0$ and $x + 1 = 12t_2$, $y - 7 = 6t_2$, $z - 5 = 3t_2$ with $t_1, t_2 \in \mathbb{R}$ intersect, and find the point of intersection.

Solution: The line L_1 has a direction vector $\langle 4, 1, 0\rangle$ and the line L_2 has a direction vector $\langle 12, 6, 3\rangle$. The 2 direction vectors are not parallel (it is not possible to express one vector as a multiple of the other one).

An intersection point must satisfy both equations of L_1 and L_2. From the equations of L_1, we know that all points on L_1 must have $z = 1$. Using $z = 1$ and the equation for the $z-$coordinates of L_2 points $z - 5 = 3t_2$, we can characterize in a unique way the value for $t_2 :\ z = 1 = 5 + 3t_2$ then $t_2 = -4/3$. Using $t_2 = -4/3$ in L_2 equations, we obtain the following point $(-1 + 12 \times (-4/3), 7 + 6 \times (-4/3), 1) = (-17, -1, 1)$ which belongs to L_2. Check if this point $(-17, -1, 1)$ belongs to L_1 (satisfies the L_1 equations):

$$\begin{aligned} -17 &= 3 + 4t_1 && \Rightarrow t_1 = -5 \\ -1 &= 4 + t_1 && \Rightarrow t_1 = -5 \\ 1 &= 1 \end{aligned}$$

Using $t_1 = -5$ in L_1 equations, we find $(-17, -1, 1)$. Therefore, $(-17, -1, 1)$ is a point of both lines L_1 and L_2. Since the two lines are not parallel, the two lines must intersect and the intersection point is $(-17, -1, 1)$.

4. Solve the linear system

$$\mathbf{AX} = \mathbf{B}$$

where $\mathbf{B}^T = [0, 0, 0]$ and

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{bmatrix}$$

Specify the method used to solve the linear system.

Solution: The augmented matrix corresponding to the homogeneous system is

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 2 & -3 & -1 & -4 & 0 \\ 3 & -5 & -1 & -1 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 1 & -1 & -10 & 0 \\ 0 & 1 & -1 & -10 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 0 & 3 & 0 \\ 0 & 1 & -1 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R_1 \to R_1 + 2R_2$$

The above matrix is the Row Echelon Form of the augmented matrix. If the Gaussian elimination method is used, stop the elementary row operations here and proceed to the back-substitutions in the system:

$$x_1 - 2x_2 + 3x_3 = 0$$
$$x_2 - x_3 - 10x_4 = 0$$

to find the solution. Otherwise, if Gauss-Jordan elimination is used, do the last elementary row operation ($R_1 \to R_1 + 2R_2$) to obtain the Reduced Row Echelon Form of the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 23 & 0 \\ 0 & 1 & -1 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The solution $\mathbf{X}^T = (x_1, x_2, x_3, x_4)$ is

$$\begin{aligned} x_1 &= 2s + 23t, \\ x_2 &= s + 10t, \\ x_3 &= s, \quad s \in \mathbb{R}, \\ x_4 &= t, \quad t \in \mathbb{R}, \end{aligned}$$

or in basic solutions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 23 \\ 10 \\ 0 \\ 1 \end{pmatrix} t, \quad s, t \in \mathbb{R}.$$

5. Compute the determinant of

   (a) $\mathbf{A} = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{bmatrix}$

   (b) $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -3 & 4 \\ 0 & 2 & -2 & 3 \\ 1 & 1 & -4 & -2 \end{bmatrix}$

   (c) $\mathbf{A}^5$ when $\mathbf{A}$ is defined as in (b).

   (d) $-\mathbf{A}$ when $\mathbf{A}$ is defined as in (b).

   (e) $\mathbf{A}^T$ when $\mathbf{A}$ is defined as in (b).

Solution:

(a) $\mathbf{A}$ is not a square matrix. The determinant of $\mathbf{A}$ does not exist.

(b) Before computing the determinant of $\mathbf{A}$ we will use row elementary operations on the matrix $\mathbf{A}$ to transform it to a triangular matrix.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & -3 & 4 \\ 0 & 2 & -2 & 3 \\ 1 & 1 & -4 & -2 \end{bmatrix} R_2 \to R_2 - 2R_1,\ R_4 \to R_4 - R_1$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 3 \\ 0 & 1 & -3 & -4 \end{bmatrix} R_3 \to R_3 - 2R_2,\ R_4 \to R_4 - R_2$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -2 & -4 \end{bmatrix} R_4 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Define $\mathbf{B} = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$.

To obtain the upper-triangular matrix $\mathbf{B}$ from $\mathbf{A}$, we used 4 additions of a nonzero multiple of one row to another row (operations that do not affect the determinant) and a single row interchange (operation that multiplies the determinant by -1). Hence, $\det(\mathbf{A}) = -\det(\mathbf{B})$.

As $\mathbf{B}$ is a upper-triangular matrix, the determinant of $\mathbf{B}$ is the product of the diagonal entries:

$$\det(\mathbf{B}) = 1 \times 1 \times (-2) \times 3 = -6.$$

Therefore, $\det(\mathbf{A}) = 6$.

(c) $\mathbf{A}^5 = \mathbf{A} \times \mathbf{A} \times \mathbf{A} \times \mathbf{A} \times \mathbf{A}$. As the determinant of a product is the product of the determinant:

$$\det(\mathbf{A}^5) = \det(\mathbf{A})^5 = 6^5.$$

(d) The matrix $\mathbf{A}$ is a 4×4 matrix, $\det(-\mathbf{A}) = (-1)^4 \det(\mathbf{A}) = 6$.

(e) $\det(\mathbf{A}^{\mathbf{T}}) = \det(\mathbf{A}) = 6$.

6. Find all values of λ for which $\det(\mathbf{A}) = 0$ for $\mathbf{A} = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$.

Solution: Compute $\det(\mathbf{A})$:

$$\det(\mathbf{A}) = (\lambda - 4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix} = (\lambda - 4)(\lambda(\lambda - 1) - 6) = (\lambda - 4)(\lambda^2 - \lambda - 6)$$

To have $\det(\mathbf{A}) = 0$, we need to find the roots of $P(\lambda) = (\lambda - 4)(\lambda^2 - \lambda - 6) = 0$. The 3 roots of $P(\lambda)$ are

$$\lambda_1 = 4$$
$$\lambda_2 = \frac{1 - \sqrt{1 + 24}}{2} = -2$$
$$\lambda_3 = \frac{1 + \sqrt{1 + 24}}{2} = 3$$

Values of λ for which $\det(\mathbf{A}) = 0$ are -2, 3 and 4.

7. Consider the following linear system

$$\begin{aligned} 2x - y + 2z &= 2, \\ x - y - z &= 1, \\ 4x + 2y - z &= 0. \end{aligned}$$

(a) What is the rank of the augmented matrix of the linear system?

(b) What is the rank of the coefficient matrix of the linear system?

(c) By using Gauss-Jordan elimination, solve the linear system.

Solution: Consider the nonhomogeneous system $\mathbf{AX} = \mathbf{B}$. Transform the augmented matrix $[\mathbf{A}|\mathbf{B}]$ to its Reduced Row Echelon Form:

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & 2 \\ 1 & -1 & -1 & 1 \\ 4 & 2 & -1 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 2 & -1 & 2 & 2 \\ 4 & 2 & -1 & 0 \end{array}\right] \quad R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 4R_1$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 6 & 3 & -4 \end{array}\right] \quad R_1 \to R_1 + R_2,\ R_3 \to R_3 - 6R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & -21 & -4 \end{array}\right] \quad R_3 \to -R_3/21$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 4/21 \end{array}\right] \quad R_1 \to R_1 - 3R_3,\ R_2 \to R_2 - 4R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9/21 \\ 0 & 1 & 0 & -16/21 \\ 0 & 0 & 1 & 4/21 \end{array}\right]$$

(a) There are 3 leading 1's in the Reduced Row Echelon Form of the augmented matrix, the rank of the augmented matrix is 3.

(b) The 3 leading 1's are to the left of the vertical bar (the part corresponding to the coefficient matrix), so the rank of the coefficient matrix is 3.

(c) The solution is

$$\begin{aligned} x &= 9/21, \\ y &= -16/21, \\ z &= 4/21. \end{aligned}$$

8. Consider the following linear system

$$\begin{aligned} 2x_1 + 2x_2 - x_3 + x_5 &= 0, \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0, \\ x_1 + x_2 - 2x_3 - x_5 &= 0, \\ x_3 + x_4 + x_5 &= 0. \end{aligned}$$

(a) By using Gauss-Jordan elimination, solve the linear system.

(b) Write your solution(s) using the basic solution(s).

(c) Is the trivial solution a solution of this linear system? Explain your answer.

Solution: Transform the augmented matrix to its Reduced Row Echelon Form.

$$\left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \quad R_2 \to R_2 + R_1,\ R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \quad R_2 \to -R_2/3,\ R_3 \to R_3/3$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_4$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2, R_1 \to R_1 + 2R_2$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \quad R_4 \to R_4 + R_3,\ R_2 \to R_2 + R_3,\ R_1 \to R_1 + 2R_3$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_2 \to -R_2$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) The system has $n = 5$ unknowns, the rank of the matrix is $r = 3$, the solution will be a $n-r=5-3=2-$parameter family. The solutions are

$$\begin{aligned} x_1 &= -s-t \\ x_2 &= s \\ x_3 &= -t \\ x_4 &= 0 \\ x_5 &= t \end{aligned}$$

with $s,t \in \mathbb{R}$.

(b) Using basic solutions, the solutions are expressed as follows

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} t, \quad s,t \in \mathbb{R}. \tag{1}$$

(c) The trivial solution $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ is always solution to a homogeneous system. For instance, take $s=t=0$ in (1), the trivial solution is obtained.