The **determinant** of a square matrix $A \in \mathcal{M}_n$ is denoted $\det(A)$ or $|A|$. It is a **scalar**.

Definition

The determinant of a 1×1 matrix (a matrix with a single entry) is the entry itself.

Definition

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then $\det(A) = |A| = ad - bc$.

Definition (Minor)

Let $A \in \mathcal{M}_n$ be a square matrix. The **minor** m_{ij} of the $(i,j)^{\mathrm{th}}$ entry is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.

Definition (Cofactor)

Let $A \in \mathcal{M}_n$ be a square matrix. The **cofactor** c_{ij} of the $(i,j)^{\mathrm{th}}$ entry is given by $c_{ij} = (-1)^{i+j} m_{ij}$, where m_{ij} is the minor of the $(i,j)^{\mathrm{th}}$ entry.

Definition (Determinant)

Let $A \in \mathcal{M}_n$ be a square matrix. The determinant of A is the scalar obtained by summing the products of the entries on any row or column and their cofactors, i.e., for some $i = 1, \ldots, n$,

$$\det(A) = |A| = \sum_{k=1}^{n} a_{ik} c_{ik},$$

or for some $j = 1, \ldots, n$,

$$\det(A) = |A| = \sum_{k=1}^{n} a_{kj} c_{kj}.$$

Theorem

Let $A, B \in \mathcal{M}_n$ be square matrices. Then

- *If A has a row or column of zeros, then $|A| = 0$.*
- *If A is diagonal or (upper or lower) triangular, then $|A| = a_{11}a_{22}\cdots a_{nn}$.*
- *If two rows (or two columns) of A are interchanged, then the determinant of A is multiplied by -1.*
- *If all entries in a row (or in a column) of A are multiplied by a scalar k, then the new matrix has determinant $k|A|$. In particular, $|kA| = k^n|A|$.*
- *If a multiple of one row (resp. column) of A is added to another row (resp. column) of A, then the resulting matrix has the same determinant as A.*
- $|AB| = |A|\,|B|$.
- *If A is invertible, then $|A^{-1}| = |A|^{-1}$.*

Theorem (Cramer's rule)

Consider the linear system $AX = B$ *with* $A \in \mathcal{M}_n$ *a square matrix such that* $|A| \neq 0$, $X = (x_1, \ldots, x_n)^T$ *and* $B = (b_1, \ldots, b_n)^T$ *column vectors. Then for* $i = 1, \ldots, n$,

$$x_i = \frac{|A_i|}{|A|},$$

where A_i *is the matrix obtained by replacing column* i *in* A *by* B.

Theorem

A homogeneous system of n linear equations in n unknowns, $AX = 0$ has only the trivial solution if and only if $|A| \neq 0$; it has nontrivial solutions if and only if $|A| = 0$.

Theorem

A nonhomogeneous system of n linear equations in n unknowns, $AX = B$ has a unique solution if and only if $|A| \neq 0$; it has either no solutions or an infinity of solutions if and only if $|A| = 0$.