Definition

Let $A \in \mathcal{M}_n$ be a square matrix. Then A is **invertible** (or **nonsingular**) if there exists a matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I_n.$$

If no such matrix exists, A is **singular**. When it exists, the inverse of a given matrix is unique.

Theorem

Let $A, B \in \mathcal{M}_n$ be invertible. Then

$$(AB)^{-1} = B^{-1}A^{-1}$$

and

$$(A^T)^{-1} = (A^{-1})^T$$

Theorem (Direct method for inversion)

Let $A \in \mathcal{M}_n$. If A is invertible, then elementary row operations on the augmented matrix $[A|I_n]$ eventually lead to the augmented matrix $[I_n|A^{-1}]$.

Definition (Adjoint matrix)

Let $A \in \mathcal{M}_n$ be a square matrix. The **matrix of cofactors** is the matrix $C \in \mathcal{M}_n$ with entries the corresponding cofactors of A. The **adjoint** (matrix) of A, $\mathrm{adj}A$, is then

$$\mathrm{adj}A = C^T.$$

Theorem

Let $A \in \mathcal{M}_n$. If $|A| \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{|A|}\text{adj}A.$$

Theorem

Let $A \in \mathcal{M}_n$ be a square matrix. A has an inverse if and only if $|A| \neq 0$.

Theorem

Let $A \in \mathcal{M}_n$ and X, B be two column vectors with n entries. If $|A| \neq 0$, then the linear system $AX = B$ has the unique solution $X = A^{-1}B$.

Theorem

Let $A \in \mathcal{M}_n$ be a square matrix, X, B be column vectors with n entries. The following are equivalent.

1. *A is invertible.*
2. $|A| \neq 0$.
3. *The reduced row echelon form of A is I_n.*
4. *A has rank n.*
5. *$AX = B$ has the unique solution $X = A^{-1}B$.*
6. *$AX = \mathbf{0}$ has only the trivial solution $X = \mathbf{0}$.*
7. *A has no eigenvalue equal to 0.*