Systems of linear equations

Definition

A *linear equation* in n variables/unknowns $x_1, \dots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, \dots, a_n$ and b are constants.

Definition

A finite set of linear equations in $x_1, \dots, x_n$ is called *a system of linear equations* or a *linear system*. An *arbitrary system* of m linear equations in n unknowns has the form:

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \cdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

Elementary operations for linear systems

There are three types of *elementary operations* for linear systems:

- Multiply an equation through by a nonzero constant.
- Interchange two equations.
- Add a nonzero multiple of one equation to another.

Theorem

If elementary operations are performed on a system of linear equations, the resulting system is equivalent to the original system; both systems have the same solutions.

Definition

The *augmented matrix* of the linear system

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

is

$$\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right].$$

Elementary row operations

Elementary operations for augmented matrices

These three operations correspond to the following *elementary row operations* for the augmented matrix:

- Multiply a row through by a nonzero constant.
- Interchange two rows.
- Add a nonzero multiple of one row to another row.

Row Echelon Form (REF)

Definition

A matrix is said to be in *row echelon form* (REF) if it satisfies the following properties.

(i) The leftmost nonzero entry in every row is a 1 (called a leading 1).

(ii) The leading 1 for each row is to the left of the leading 1 in every row below it.

(iii) The entries below every leading 1 are all 0's.

(iv) Any rows of all 0's are below all rows that have leading 1.

Note that in the above definition, Property (iii) is actually a result of (i) and (ii).

Gaussian elimination

The following procedure/algorithm, called *Gaussian elimination*, takes an augmented matrix to REF:

Step 1. Find the left column that has a nonzero entry in it. Choose a nonzero entry in this column and interchange rows, if necessary, to put this entry in the top row.

Step 2. Divide the top row by the nonzero entry found in Step 1 so that the leftmost entry in the row is 1 (this is the leading 1).

Step 3. Add suitable multiples of the top row to the rows below it to create 0's below the leading 1.

Step 4. Pretend the top row is not there and repeat Steps 1, 2, and 3 on the remaining rows.

REFs of a matrix are not unique.

Reduced Row Echelon Form (RREF)

Definition

A matrix is said to be in *reduced row echelon form* (RREF) if it satisfies the Properties (i)–(iv) in the definition of the REF and the following property:

(v) The entries above every leading 1 are all 0's.

Theorem

An augmented matrix has exactly one RREF.

Gauss-Jordan elimination

The following procedure/algorithm, called *Gauss-Jordan elimination*, takes an augmented matrix to RREF:

Steps 1-4. Same as Gaussian elimination. After getting an REF, perform the following Step 5.

Step 5. Beginning with the last nonzero row and working upward, add suitable multiples of the each row to the rows above to create 0's above the leading 1's.

Definition

The *rank*, r, of a matrix is the number of leading 1's in its RREF.

Consider the linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Define the *coefficient matrix* A of this system, the matrix of unknowns X, and the matrix of coefficients B by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

The matrix form of the linear system is

$$AX = B.$$

Definition

The system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

of linear equations is said to be **homogeneous** if all the b_i are equal to zero. If at least one of the b_i is nonzero, the system is said to be **nonhomogeneous**.

Three possibilities of solutions of a nonhomogeneous system:

(i) No solution (when the system is inconsistent)

(ii) Exactly one solution (when the system is consistent and $r = n$, where r is the rank of the coefficient matrix A)

(iii) Infinitely many solutions (when the system is consistent and $r < n$)

Two possibilities of solutions of a homogeneous system:

(i) Exactly one solution, the trivial solution (when $r = n$)

(ii) Infinitely many solutions (when $r < n$)

A homogeneous system is always consistent, because $x_1 = x_2 = \cdots = x_n = 0$ is always a solution, called the *trivial solution*.