Matrices

Definition

An $m \times n$ matrix is a rectangular array of mn numbers arranged in m rows and n columns. We write the set of $m \times n$ matrix $\mathcal{M}_{mn}$ and for a matrix $A \in \mathcal{M}_{mn}$ (which we also write $A_{m\times n}$),

$$A = [a_{ij}] = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

Special matrices

- A matrix A is **square** if $m = n$, in which case we denote $A \in \mathcal{M}_n$.
- A square matrix is **diagonal** if its only nonzero entries are (perhaps) on the diagonal, i.e., if $a_{ij} = 0$ whenever $i \neq j$.
- A diagonal matrix with all 1's on the diagonal is called the **identity** matrix (of order n).
- A square matrix is **upper** (resp. **lower**) triangular if all entries below (resp. above) the diagonal are zero, i.e., $a_{ij} = 0$ when $i > j$ (resp. $i < j$).

Basic operations

- Equality: $A = B$ iff $a_{ij} = b_{ij}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.
- Addition: $A + B = [a_{ij} + b_{ij}]$.
- Subtraction: $A - B = [a_{ij} - b_{ij}]$.
- Scalar multiplication: $cA = [ca_{ij}]$ for all $c \in \mathbb{R}$ or $\mathbb{C}$.
- Transpose: The **transpose** of $A = [a_{ij}]$ is $A^T = [a_{ji}]$.

Properties of basic operations

$$
\begin{aligned}
A + B &= B + A && \text{Commutativity of adition} \\
A + (B + C) &= (A + B) + C && \text{Associativity of adition} \\
\lambda(\mu A) &= (\lambda\mu)A \\
(\lambda + \mu)A &= \lambda A + \mu A \\
\lambda(A + B) &= \lambda A + \lambda B \\
A + \mathbf{0} &= A \\
(A^T)^T &= A
\end{aligned}
$$

Multiplication of matrices

Let $A \in \mathcal{M}_{mp}$ and $B \in \mathcal{M}_{pn}$. The matrix $C = AB$ has dimension $m \times n$ and entries given by

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

Matrix multiplication has the following properties, where A, B, C have dimensions which make these operations possible:

- $A(BC) = (AB)C$ [Associativity of multiplication]
- $(A + B)C = AC + BC$ [Distributivity of multiplication over addition]
- $A(B + C) = AB + AC$ [Distributivity of multiplication over addition]
- $\mathbf{0}A = \mathbf{0}$ and $A\mathbf{0} = \mathbf{0}$
- $(\alpha A)(\beta B) = (\alpha\beta)AB$
- $A_{m\times n}I_n = I_m A_{m\times n} = A_{m\times n}$
- The following is extremely important:

$$(AB)^T = B^T A^T$$

- Matrix multiplication **is not** commutative, i.e., in general, $AB \neq BA$