Polynomials

Definition

A polynomial in x of degree n, where $n \geq 0$ is an integer, is

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n \neq 0$ and $a_{n-1}, \cdots, a_0$ are constants.

Definition

A polynomial equation is an equation of the form:

$$P_n(x) = 0$$

or

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

Remainder Theorem

When a polynomial $P_n(x)$ is divided by $bx - a$, the remainder is $P_n(a/b)$. $P_n(x)$ can be expressed in the form

$$P_n(x) = (bx - a)Q_{n-1}(x) + P_n(a/b)$$

where $Q_{n-1}(x)$ is a polynomial of degree $n - 1$.

Factor Theorem

$bx - a$ is a factor of $P_n(x)$ if and only if $P_n(a/b) = 0$.

Fundamental theorem of Algebra I

Every polynomial of degree $n \geq 1$ has exactly n linear factors (which may not all be different).

Fundamental theorem of Algebra II

Every polynomial of degree $n \geq 1$ has exactly n zeros (counting multiplicities).

Theorem

If z is a complex zero of a polynomial with **real** coefficients, then so also is $\bar{z}$.

Theorem

Every real polynomial can be factored into the product of real linear and irreducible real quadratic factors.

Irreductible factor

An irreducible factor is a quadratic factor that cannot be factored into real linear factors.

Rational root theorem

Suppose that $r = p/q$ is a rational root (in lowest terms) of a polynomial equation $a_n x^n + \cdots + a_1 x + a_0 = 0$ with integer coefficients, and $a_0 \neq 0$. Then p divides a_0 and q divides a_n.

Descartes' rule theorem

When $P(x)$ is a polynomial with real coefficients written in descending (or ascending) powers of x

a) the number of positive (real) zeros (counting multiplicities) is equal to the number of sign changes in the coefficients, or equal to the number of sign changes decreased by an even integer;

b) the number of negative (real) zeros (counting multiplicities) is equal to the number of sign changes in the coefficients of $P(-x)$, or equal to the number of sign changes decreased by an even integer.

Bounds theorem

If x is a zero of the n^{th} degree polynomial

$$P(x) = a_n x^n + \cdots + a_1 x + a_0,$$

then

$$|x| < \frac{M}{|a_n|} + 1$$

where $M = maximum\{|a_{n-1}|, |a_{n-2}|, \cdots, |a_0|\}$.